Lumping-based equivalences in Markovian automata and applications to product-form analyses

Andrea Marin and Sabina Rossi

DAIS - Università Ca' Foscari, Venezia, Italy
{marin,srossi}@dais.unive.it

Abstract. The analysis of models specified with formalisms like Markovian process algebras or stochastic automata can be based on equivalence relations among the states. In this paper we introduce a relation called *exact equivalence* that, differently from most aggegation approaches, induces an exact lumping on the underlying Markov chain instead of a strong lumping. We prove that this relation is a congruence for Markovian process algebras and stochastic automata whose synchronisation semantics can be seen as the master/slave synchronisation of the Stochastic Automata Networks (SAN). We show the usefulness of this relation by proving that the class of quasi-reversible models is closed under exact equivalence. Quasi-reversibility is a pivotal property to study product-form models, i.e., models whose equilibrium behaviour can be computed very efficiently without the problem of the state space explosion. Hence, exact equivalence turns out to be a theoretical tool to prove the product-form of models by showing that they are exactly equivalent to models which are known to be quasi-reversible.

1 Introduction

Stochastic modelling plays an important role in computer science since it is widely used for performance evaluation and reliability analysis of software and hardware architectures, including telecommunication systems. In this context, Continuous Time Markov Chains (CTMCs) are the underlying stochastic processes of the models specified with many formalisms such as Stochastic Petri nets [22], Stochastic Automata Networks (SAN) [24], queueing networks [3] and a class of Markovian process algebras (MPAs), e.g., [14, 12]. The aim of these formalisms is to provide a high-level description language for complex models and automatic analysis methods. Modularity in the model specification is an important feature of both MPAs and SANs that allows for describing large systems in terms of cooperations of simpler components. Nevertheless, one should notice that a modular specification does not lead to a modular analysis, in general. Thus, although the intrinsic compositional properties of such formalisms are extremely helpful in the specification of complex systems, in many cases carrying out an exact analysis for those models (e.g., those required by quantitative model checking) may be extremely expensive from a computational point of view.

The introduction of equivalence relations among quantitative models is an important formal approach to comparing different systems and also improving the efficiency of some analysis. Indeed, if we can prove that a model P is in some sense equivalent to Q and Q is much simpler than P, then we can carry out an analysis of the simplest component to derive the properties of the original one.

Bisimulation based relations on stochastic systems inducing the notions of ordinary (or strong) and exact lumpability for the underlying Markov chains have been studied in [7, 2, 14, 10, 26]. In this paper, we apply this idea by introducing the notion of *exact equivalence* on the states of synchronising stochastic automata and study its compositionality properties. Exact equivalence is a congruence for the synchronisation semantics that we consider, i.e., it is preserved by the synchronising operator. Moreover, we prove that an exact equivalence relation among the states of a non-synchronising automaton induces an exact lumping on its underlying CTMC. This is opposed to the usual notions of bisimulation-based equivalences previously introduced in the literature that induce a strong lumping [18] on the underlying CTMC [8, 2, 6, 14, 20]. Interestingly, we show that an exact equivalence over a non-synchronising stochastic automata induces a *strong lumping* on the time-reversed Markov chain underlying the model. This important observation, allows us to prove that exact equivalence preserves the quasi-reversibility property [17] defined for stochastic networks. Quasi-reversibility is one of the most important and widely used characterisation of product-form models, i.e., models whose equilibrium distribution can be expressed as the product of functions depending only on the local state of each component. Informally, we can say that product-forms project the modularity in the model definition to the model analysis, thus drastically reducing the computational costs of the derivation of the quantitative indices. Basically, a synchronisation of quasi-reversible components whose underlying chain is ergodic has a product-form solution, meaning that one can check the quasi-reversibility modularly for each component, without generating the whole state space.

In this paper we provide a new methodology to prove (disprove) that a stochastic automaton is quasi-reversible by simply showing that it is exactly equivalent to another model which is known to be (to be not) quasi-reversible. In practice, this approach can be useful because proving the quasi-reversibility of a model may be a hard task since it requires one to reverse the underlying CTMC and check some conditions on the reverse process, see, e.g., [17, 11]. Conversely, by using exact equivalence, one can prove or disprove the quasi-reversibility property by considering only the forward model, provided that it is exactly equivalent to another (simpler) quasi-reversible model known in the wide literature of product-forms. Moreover, while automatically proving quasi-reversibility is in general unfeasible, checking the exact equivalence between two automata can be done algoritmically by exploiting a partition refinement strategy, similar to that of Paige and Tarjan's algorithm for bisimulation [23].

The paper is structured as follows. Section 2 introduces the notation and recalls the basic definitions on Markov chains. In Section 3 we give the definition of stochastic automata and specify their synchronisation semantics. Section 4 presents the definition of quasi-reversibility for stochastic automata. Exact equivalence is introduced in Section 5 and the fact that it preserves quasi-reversibility is proved. Finally, Section 6 concludes the paper.

2 Preliminaries

Let $X(t)$ be a stochastic process taking values into a state space $\mathcal{S}$ for $t \in \mathbb{R}^+$. $X(t)$ is said *stationary* if $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}^+$. Moreover, $X(t)$ satisfies the *Markov property*, and it is called *Markov process*, if the conditional (on both past and present states) probability distribution of its future behaviour is independent of its past evolution until the present state. A Continuous-Time Markov Chain (CTMC) is a Markov process with a discrete state space $\mathcal{S}$.

A CTMC $X(t)$ is said to be *time-homogeneous* if the conditional probability $P(X(t+\tau) = s \mid X(t) = s')$ does not depend upon t, and is *irreducible* if every state in $\mathcal{S}$ can be reached from every other state. A state in a Markov process is called *recurrent* if the probability that the process will eventually return to the same state is one. A recurrent state is called *positive-recurrent* if the expected number of steps until the process returns to it is finite. A CTMC is *ergodic* if it is irreducible and all its states are positive-recurrent. For finite Markov chains, irreducibility is sufficient for ergodicity.

An ergodic CTMC possesses an *equilibrium* (or *steady-state*) *distribution*, that is the *unique* collection of positive numbers $\pi(s)$ with $s \in \mathcal{S}$ such that

$$\lim_{t \to \infty} P(X(t) = s \mid X(0) = s') = \pi(s)\,.$$

The transition rate between two states s and s' is denoted by $q(s, s')$, with $s \neq s'$. The infinitesimal generator matrix $\mathbf{Q}$ of a Markov process is such that the $q(s, s')$'s are the off-diagonal elements while the diagonal elements are formed as the negative sum of the extra diagonal elements of each row. Any non-trivial vector of real numbers $\boldsymbol{\mu}$ satisfying the system of global balance equations (GBEs)

$$\boldsymbol{\mu}\mathbf{Q} = \mathbf{0} \tag{1}$$

is called *invariant measure* of the CTMC. For irreducible CTMCs, if $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are both invariant measures of the same chain, then there exists a constant $k > 0$ such that $\boldsymbol{\mu}_1 = k\boldsymbol{\mu}_2$. If the CTMC is ergodic, then there exists a unique invariant measure $\boldsymbol{\pi}$ whose components sum to unity, i.e., $\sum_{s \in \mathcal{S}} \pi(s) = 1$. In this case $\boldsymbol{\pi}$ is the equilibrium or steady-state distribution of the CTMC.

It is well-known that the solution of system (1) is often unfeasible due to the large number of states of the CTMC underlying the model of a real system. The analysis of an ergodic CTMC in equilibrium can be greatly simplified if it satisfies the property that when the direction of time is reversed the stochastic behaviour of the process remains the same.

Given a stationary CTMC $X(t)$ with $t \in \mathbb{R}^+$, we call $X(\tau - t)$ its reversed process. In the following we denote by $X^R(t)$ the reversed process of $X(t)$.

It can be shown that $X^R(t)$ is also a stationary CTMC [17]. We say that $X(t)$ is *reversible* if it is stochastically identical to $X^R(t)$, i.e., the process $(X(t_1), \ldots, X(t_n))$ has the same distribution as $(X(\tau - t_1), \ldots, X(\tau - t_n))$ for all $t_1, \ldots, t_n, \tau \in \mathbb{R}^+$ [17].

For a stationary Markov process there exists a necessary and sufficient condition for reversibility expressed in terms of the equilibrium distribution $\boldsymbol{\pi}$ and the transition rates.

Proposition 1. (Transition rates and probabilities of reversible processes [17]) *A stationary CTMC with state space $\mathcal{S}$ and infinitesimal generator $\mathbf{Q}$ is reversible if there exists a vector of positive real numbers $\boldsymbol{\pi}$ summing to unity, such that for all $s, s' \in \mathcal{S}$ with $s \neq s'$,*

$$\pi(s)q(s,s') = \pi(s')q(s',s) \,.$$

In this case $\boldsymbol{\pi}$ is the equilibrium distribution of the chain.

The *reversed process* $X^R(t)$ of a Markov process $X(t)$ can always be defined even when $X(t)$ is not reversible. In [17, 11] the authors show that $X^R(t)$ is a CTMC and its transition rates are defined according to the following proposition.

Proposition 2. (Transition rates of reversed processes [11]) *Given the stationary CTMC $X(t)$ with state space $\mathcal{S}$ and infinitesimal generator $\mathbf{Q}$, the transition rates of the reversed process $X^R(t)$, forming its infinitesimal generator $\mathbf{Q}^R$, are defined as follows: for all $s, s' \in \mathcal{S}$,*

$$q^R(s',s) = \frac{\mu(s)}{\mu(s')} q(s,s') \,, \tag{2}$$

where $q^R(s',s)$ denotes the transition rate from s' to s in the reversed process and $\boldsymbol{\mu}$ is an invariant measure of $X(t)$.

The forward and the reversed processes share all the invariant measures and in particular they possess the same equilibrium distribution $\boldsymbol{\pi}$.

In the following, for a given CTMC with state space $\mathcal{S}$ and for any state $s \in \mathcal{S}$ we denote by $q(s)$ (resp., $q^R(s)$) the quantity $\sum_{s' \in \mathcal{S}, s \neq s'} q(s,s')$ (resp., $\sum_{s' \in \mathcal{S}, s \neq s'} q^R(s,s')$).

In the context of performance and reliability analysis, the notion of *lumpability* is used for generating an aggregated Markov process that is smaller than the original one but allows one to determine exact results for the original process. More precisely, the concept of lumpability can be formalized in terms of equivalence relations over the state space of the Markov chain. Any such equivalence induces a *partition* on the state space of the Markov chain and aggregation is achieved by clustering equivalent states into macro-states, thus reducing the overall state space. In general, when a CTMC is aggregated the resulting stochastic process will not have the Markov property. However, if the partition can be shown to satisfy the so called *strong* lumpability condition [18, 1], the Markov property is preserved and the equilibrium solution of the aggregated process may be used to derive an exact solution of the original one.

Strong lumpability has been introduced in [18] and further studied in [9, 27].

Definition 1. (Strong lumpability) *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. We say that $X(t)$ is* strongly lumpable *with respect to $\sim$ (resp., $\sim$ is a* strong lumpability *for $X(t)$) if $\sim$ induces a partition on the state space of $X(t)$ such that for any equivalence class $S_i, S_j \in \mathcal{S}/\sim$ with $i \neq j$ and $s, s' \in S_i$,*

$$\sum_{s'' \in S_j} q(s, s'') = \sum_{s'' \in S_j} q(s', s'') .$$

Thus, an equivalence relation over the state space of a Markov process is a strong lumpability if it induces a partition into equivalence classes such that for any two states within an equivalence class their aggregated transition rates to any other class are the same. Notice that every Markov process is strongly lumpable with respect to the identity relation, and so it is the trivial relation having only one equivalence class.

A probability distribution $\boldsymbol{\pi}$ is *equiprobable* with respect to a partition of the state space $\mathcal{S}$ of an ergodic Markov process if for all the equivalence classes $S_i \in \mathcal{S}/\sim$ and for all $s, s' \in S_i$, $\pi(s) = \pi(s')$.

In [25] the notion of exact lumpability is introduced as a sufficient condition for a distribution to be equiprobable with respect to a partition.

Definition 2. (Exact lumpability) *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. We say that $X(t)$ is* exactly lumpable *with respect to $\sim$ (resp., $\sim$ is an exact lumpability for $X(t)$) if $\sim$ induces a partition on the state space of $X(t)$ such that for any $S_i, S_j \in \mathcal{S}/\sim$ and $s, s' \in S_i$,*

$$\sum_{s'' \in S_j} q(s'', s) = \sum_{s'' \in S_j} q(s'', s') .$$

An equivalence relation is an exact lumpability if it induces a partition on the state space such that for any two states within an equivalence class the aggregated transition rates into such states from any other class are the same.

The proof of next proposition is given in [25].

Proposition 3. *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. If $X(t)$ is* exactly lumpable *with respect to $\sim$ (resp., $\sim$ is an exact lumpability for $X(t)$) then for all $s, s' \in \mathcal{S}$ such that $s \sim s'$, $\mu(s) = \mu(s')$, where $\boldsymbol{\mu}$ is an invariant measure for $X(t)$.*

3 Stochastic Automata

Many high-level specification languages for stochastic discrete-event systems are based on *Markovian process algebras* [14, 7, 13] that are characterized by powerful composition operators and timed actions whose delay is governed by independent random variables with a continuous-time exponential distribution. The expressivity of such languages allows the development of well-structured specifications and efficient analyses of both qualitative and quantitative properties in a single framework. Their semantics is given in terms of stochastic automata, an extension of labelled automata with clocks that often are exponentially distributed random variables. In this paper we consider stochastic concurrent automata with an underlying continuous time Markov chain as common denominator of a wide set of Markovian stochastic process algebra. Stochastic automata are equipped with a *composition operation* by which a complex automaton can be constructed from simpler components. Our model draws a distinction between *active* and *passive* action types, and in forming the composition of automata only active/passive synchronisations are permitted. An analogue semantics is proposed for Stochastic Automata Networks in [24].

Definition 3. (Stochastic Automaton (SA)) *A stochastic automaton P is a tuple $(\mathcal{S}_P, \mathcal{A}_P, \mathcal{P}_P, \rightsquigarrow_P, q_P)$ where*

- $\mathcal{S}_P$ *is a denumerable set of states called* state space *of P,*
- $\mathcal{A}_P$ *is a denumerable set of* active *types,*
- $\mathcal{P}_P$ *is a denumerable set of* passive *types,*
- τ *denotes the* unknown *type,*
- $\rightsquigarrow_P \subseteq (\mathcal{S}_P \times \mathcal{S}_P \times \mathcal{T}_P)$ *is a transition relation where $\mathcal{T}_P = (A_P \cup \mathcal{P}_P \cup \{\tau\})$ and for all $s \in \mathcal{S}_P$, $(s, s, \tau) \notin \rightsquigarrow_P$,*[1]
- q_P *is a function from $\rightsquigarrow_P$ to $\mathbb{R}^+$ such that $\forall s_1 \in \mathcal{S}_P$ and $\forall a \in \mathcal{P}_P$, $\sum_{s_2:(s_1,s_2,a)\in\rightsquigarrow_P} q_P(s_1, s_2, a) \leq 1$.*

In the following we denote by $\rightarrow_P$ the relation containing all the tuples of the form (s_1, s_2, a, q) where $(s_1, s_2, a) \in\rightsquigarrow_P$ and $q = q_P(s_1, s_2, a)$. We say that $q_P(s, s', a) \in \mathbb{R}^+$ is the *rate* of the transition from state s to s' with type a if $a \in \mathcal{A}_P \cup \{\tau\}$. Notice that this is indeed the apparent transition rate from s to s' relative to a. If a is passive then $q_P(s, s', a) \in (0, 1]$ denotes the *probability* that the automaton synchronises on type a with a transition from s to s'. Hereafter, we assume that $q_P(s, s', a) = 0$ whenever there are no transitions with type a from s to s'. If $s \in \mathcal{S}_P$, then for all $a \in \mathcal{T}_P$ we write $q_P(s, a) = \sum_{s'\in\mathcal{S}} q_P(s, s', a)$. Moreover we denote by $q_P(s, s') = \sum_{a\in\mathcal{T}_P} q_P(s, s', a)$ and $q_P(s) = \sum_{a\in\mathcal{T}_P} q_P(s, a)$. We say that P is *closed* if $\mathcal{P}_P = \emptyset$. We use the notation $s_1 \overset{a}{\rightsquigarrow}_P s_2$ to denote the tuple $(s_1, s_2, a) \in\rightsquigarrow_P$; we denote by $s_1 \xrightarrow{(a,r)}_P s_2$ (resp., $s_1 \xrightarrow{(a,p)}_P s_2$) the tuple $(s_1, s_2, a, r) \in\rightarrow_P$ (resp., $(s_1, s_2, a, p) \in\rightarrow_P$).

Definition 4. (CTMC underlying a closed SA) *The CTMC underlying a closed stochastic automaton P, denoted $X_P(t)$, is defined as the CTMC with state space $\mathcal{S}_P$ and infinitesimal generator matrix $\mathbf{Q}$ defined as: for all $s_1 \neq s_2 \in \mathcal{S}_P$,*

$$q(s_1, s_2) = \sum_{a,r:(s_1,s_2,a,r)\in\rightarrow_P} r\,.$$

For ergodic chains, we denote an invariant measure and the equilibrium distribution of the CTMC underlying P by $\boldsymbol{\mu}_P$ and $\boldsymbol{\pi}_P$, respectively.

[1] Notice that τ self-loops do not affect the equilibrium distribution of the CTMC underlying the automaton. Moreover, the choice of excluding τ self-loops will simplify the definition of automata synchronisation.

$$\frac{s_{p_1} \xrightarrow{(a,r)}_P s_{p_2} \quad s_{q_1} \xrightarrow{(a,p)}_Q s_{q_2}}{(s_{p_1}, s_{q_1}) \xrightarrow{(a,pr)}_{P\otimes Q} (s_{p_2}, s_{q_2})} \quad (a \in \mathcal{A}_P = \mathcal{P}_Q)$$

$$\frac{s_{p_1} \xrightarrow{(a,p)}_P s_{p_2} \quad s_{q_1} \xrightarrow{(a,r)}_Q s_{q_2}}{(s_{p_1}, s_{q_1}) \xrightarrow{(a,pr)}_{P\otimes Q} (s_{p_2}, s_{q_2})} \quad (a \in \mathcal{P}_P = \mathcal{A}_Q)$$

$$\frac{s_{p_1} \xrightarrow{(\tau,r)}_P s_{p_2}}{(s_{p_1}, s_{q_1}) \xrightarrow{(\tau,r)}_{P\otimes Q} (s_{p_2}, s_{q_1})} \qquad \frac{s_{q_1} \xrightarrow{(\tau,r)}_Q s_{q_2}}{(s_{p_1}, s_{q_1}) \xrightarrow{(\tau,r)}_{P\otimes Q} (s_{p_1}, s_{q_2})}$$

Table 1: Operational rules for SA synchronisation

We say that an automaton is *irreducible* if each state can be reached by any other state after an arbitrary number of transitions. We say that a closed automaton P is *ergodic* if its underlying CTMC is ergodic.

The synchronisation operator between two stochastic automata P and Q is defined in the style of master/slave synchronisation of SANs [24] based on the Kronecker's algebra and the active/passive cooperation used in Markovian process algebra such as PEPA [14].

Definition 5. (SA synchronisation) *Given two stochastic automata P and Q such that $\mathcal{A}_P = \mathcal{P}_Q$ and $\mathcal{A}_Q = \mathcal{P}_P$ we define the automaton $P \otimes Q$ as follows:*

- $\mathcal{S}_{P\otimes Q} = \mathcal{S}_P \times \mathcal{S}_Q$,
- $\mathcal{A}_{P\otimes Q} = \mathcal{A}_P \cup \mathcal{A}_Q = \mathcal{P}_P \cup \mathcal{P}_Q$,
- $\mathcal{P}_{P\otimes Q} = \emptyset$,
- *τ is the unknown type,*
- *$\rightsquigarrow_{P\otimes Q}$ and $q_{P\otimes Q}$ are defined according to the rules for $\rightarrow_{P\otimes Q}$ depicted in Table 1: indeed, the relation $\rightarrow_{P\otimes Q}$ contains the tuples $((s_{p_1}, s_{q_1}),(s_{p_1}, s_{q_2}), a, q)$ with $((s_{p_1}, s_{q_1}),(s_{p_1}, s_{q_2}), a) \in \rightsquigarrow_{P\otimes Q}$ and $q = q_{P\otimes Q}((s_{p_1}, s_{q_1}), (s_{p_1}, s_{q_2}), a)$.*

Given a closed stochastic automaton P we can define its reversed P^R in the style of [4], that is a stochastic automaton whose underlying CTMC $X_{P^R}(t)$ is identical to $X_P^R(t)$.

Definition 6. (Reversed SA [4]) *Let P be a closed stochastic automaton with an underlying irreducible CTMC and let $\boldsymbol{\mu}_P$ be an invariant measure. Then we define the stochastic automaton P^R reversed of P as follows:*

- $\mathcal{S}_{P^R} = \{s^R \mid s \in \mathcal{S}_P\}$
- $\mathcal{A}_{P^R} = \mathcal{A}_P$ *and* $\mathcal{P}_{P^R} = \mathcal{P}_P = \emptyset$
- $\rightsquigarrow_{P^R} = \{(s_1^R, s_2^R, a) : (s_2, s_1, a) \in \rightsquigarrow_P, a \in \mathcal{A}_P \cup \{\tau\}\}$
- $q_{P^R}(s_1^R, s_2^R, a) = \mu_P(s_2)/\mu_P(s_1) q_P(s_2, s_1, a)$

It can be easily proved that for any invariant measure (including the equilibrium distribution) $\boldsymbol{\mu}_P$ for P there exists an invariant measure $\boldsymbol{\mu}_{P^R}$ for P^R such that for all $s \in \mathcal{S}_P$ it holds $\mu_P(s) = \mu_{P^R}(s^R)$, and viceversa.

4 Quasi-Reversible Automata

In this section we review the definition of quasi-reversibility given by Kelly in [17] by using the notation of stochastic automata. In order to clarify the exposition, we introduce a closure operation over stochastic automata that allows us to assign to all the transitions with the same passive type the same rate λ.

Definition 7. (SA closure) *The closure of a stochastic automaton P with respect to a passive type $a \in \mathcal{P}_P$ and a rate $\lambda \in \mathbb{R}^+$, written $P^c = P\{a \leftarrow \lambda\}$, is the automaton defined as follows:*

- $\mathcal{S}_{P^c} = \{s^c \mid s \in \mathcal{S}_P\}$
- $\mathcal{A}_{P^c} = \mathcal{A}_P$ *and* $\mathcal{P}_{P^c} = \mathcal{P}_P \setminus \{a\}$
- $\rightsquigarrow_{P^c} = \{(s_1^c, s_2^c, b) |\ (s_1, s_2, b) \in \rightsquigarrow_P,\ a \neq b\} \cup \{(s_1^c, s_2^c, \tau) |\ (s_1, s_2, a) \in \rightsquigarrow_P\}$
-
$$q_{P^c}(s_1^c, s_2^c, b) = \begin{cases} q_P(s_1, s_2, b) & \text{if } b \neq a, \tau \\ q_P(s_1, s_2, a)\lambda + q_P(s_1, s_2, \tau) & \text{if } b = \tau \end{cases}$$

  where we assume that $q_P(s_1, s_2, b) = 0$ if $(s_1, s_2, b) \notin \rightsquigarrow_P$.

Notice that for a closure P^c of a stochastic automaton P with respect to all its passive types in $\mathcal{P}_P$ we can compute the equilibrium distribution, provided that the underlying CTMC is ergodic (see Definition 4).

Definition 8. (Quasi-reversible SA [17, 21]) *An irreducible stochastic automaton P with $\mathcal{P}_P = \{a_1, \ldots, a_n\}$ and $\mathcal{A}_P = \{b_1, \ldots b_m\}$ is quasi-reversible if*

- *for all $a \in \mathcal{P}_P$ and for all $s \in \mathcal{S}_P$, $\sum_{s' \in \mathcal{S}_P} q_P(s, s', a) = 1$*
- *for each closure $P^c = P\{a_1 \leftarrow \lambda_1\} \ldots \{a_n \leftarrow \lambda_n\}$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ there exists a set of positive real numbers $\{k_{b_1}, \ldots, k_{b_m}\}$ such that for each $s \in \mathcal{S}_{P^c}$ and $1 \leq i \leq m$*
$$k_{b_i} = \frac{\sum_{s' \in \mathcal{S}_{P^c}} \mu_{P^c}(s') q_{P^c}(s', s, b_i)}{\mu_{P^c}(s)}, \tag{3}$$
  where $\boldsymbol{\mu}_{P^c}$ denotes any non-trivial invariant measure of the CTMC underlying P^c.

Notice that in the definition of quasi-reversibility we do not require the closure of P with respect to all its passive types to originate a stochastic automaton with an ergodic underlying CTMC because we assume $\boldsymbol{\mu}_{P^c}$ to be an invariant measure, i.e., we do not require that $\sum_{s \in \mathcal{S}_{P^c}} \mu_{P^c}(s) = 1$. However, the irreducibility of the CTMC underlying the automaton ensures that all the invariant measures differ for a multiplicative constant, hence Equation (3) is independent of the choice of the invariant measure.

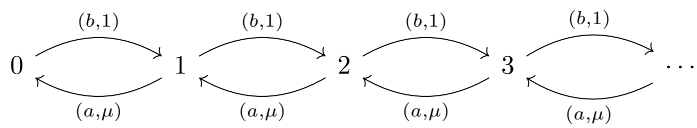

Fig. 1: Stochastic automaton underlying a Jackson's queue.

The next theorem states that a network of quasi-reversible stochastic automata exhibits a product-form invariant measure and, if the joint state space is ergodic, a product-form equilibrium distribution. For the sake of simplicity, we state the theorem for two synchronising stochastic automata although the result holds for any finite set of automata which synchronise pairwise [17, 11, 21].

Theorem 1. (Product-form solution based on quasi-reversibility) *Let P and Q be two quasi-reversible automata such that $\mathcal{A}_P = \mathcal{P}_Q$ and $\mathcal{A}_Q = \mathcal{P}_P$ and let $S = P \otimes Q$. Assume that there exists a set of positive real numbers $\{k_a : a \in \mathcal{A}_P \cup \mathcal{A}_Q\}$ such that if we define the following automata $P^c = P\{a \leftarrow k_a\}$ for each $a \in \mathcal{P}_P$ and $Q^c = Q\{a \leftarrow k_a\}$ for each $a \in \mathcal{P}_Q$ it holds that:*

$$k_a = \frac{\sum_{s' \in \mathcal{S}_{P^c}} \mu_{P^c}(s') q_{P^c}(s', s, a)}{\mu_{P'}(s)} \qquad \forall s \in \mathcal{S}_{P^c}, a \in \mathcal{A}_P$$

$$k_a = \frac{\sum_{s' \in \mathcal{S}_{Q^c}} \mu_{Q'}(s') q_{Q^c}(s', s, a)}{\mu_{Q^c}(s)} \qquad \forall s \in \mathcal{S}_{Q^c}, a \in \mathcal{A}_Q$$

Then, given the invariant measures $\boldsymbol{\mu}_{P^c}$ and $\boldsymbol{\mu}_{Q^c}$ it holds that

$$\mu_S(s_1, s_2) = \mu_{P^c}(s_1^c)\mu_{Q^c}(s_2^c)$$

is an invariant measure for all the positive-recurrent states $(s_1, s_2) \in \mathcal{S}_S$ where s_1^c and s_2^c are the states in $\mathcal{S}_{P^c}$ and $\mathcal{S}_{Q^c}$ corresponding to $s_1 \in \mathcal{S}_P$ and $s_2 \in \mathcal{S}_Q$ according to Definition 7. In this case we say that P and Q have a quasi-reversibility based product-form.

Example 1. (Product-form solution of Jackson networks) Jackson networks provide an example of models having a product-form solution. A network consists of a collection of exponential queues with state-independent probabilistic routing. Jobs arrive from the outside at each queuing station in the network according to a homogeneous Poisson process. It is well-known that the queues of Jackson networks are quasi-reversible and hence the product-form is a consequence of Theorem 1. Figure 1 shows the automaton underlying a Jackson's queue where a is an active type while b is a passive one. It is worth of notice that also the queues considered in [5, 19] are quasi-reversible. □

5 Lumpable Bisimulations and Exact Equivalences

In this section we introduce two coinductive definitions, named *lumpable bisimulation* and *exact equivalence*, over stochastic automata which provide a sufficient condition for strong and exact lumpability of the underlying CTMCs.

The lumpable bisimulation is developed in the style of Larsen and Skou's bisimulation [16]. In this section we restrict ourself to the class of irreducible stochastic automata. Let $\mathcal{S}^*$ be the set of all states of all irreducible stochastic automata and $\mathcal{T}^*$ be the set of all action types of all stochastic automata. As expected, for any $s, s' \in \mathcal{S}^*$ and for any $a \in \mathcal{T}^*$, $q(s, s', a)$ denotes $q_P(s, s', a)$ if $s, s' \in \mathcal{S}_P$ for some stochastic automaton P, otherwise $q(s, s', a)$ is equal to 0. Analogously, we write $q(s, a)$ to denote $q_P(s, a)$ when $s \in \mathcal{S}_P$ for some stochastic automaton P.

Definition 9. (Lumpable bisimulation) *An equivalence relation $\mathcal{R} \subseteq \mathcal{S}^* \times \mathcal{S}^*$ is a* lumpable bisimulation *if whenever $(s, s') \in \mathcal{R}$ then for all $a \in \mathcal{T}^*$ and for all $C \in \mathcal{S}^*/\mathcal{R}$ such that*

- *either $a \neq \tau$,*
- *or $a = \tau$ and $s, s' \notin C$,*

it holds $\sum_{s'' \in C} q(s, s'', a) = \sum_{s'' \in C} q(s', s'', a)$.

It is clear that the identity relation is a lumpable bisimulation. In [15] we proved that the transitive closure of a union of lumpable bisimulations is still a lumpable bisimulation. Hence, the maximal lumpable bisimulation, denoted $\sim_s$, is defined as the union of all the lumpable bisimulations. We say that two stochastic automata P and Q are equivalent according to the lumpable bisimulation equivalence relation, denoted $P \sim_s Q$, if there exists $s_p \in \mathcal{S}_P$ and $s_q \in \mathcal{S}_Q$ such that $(s_p, s_q) \in \sim_s$. For any stochastic automaton P, $\sim_s$ induces a partition on the state space of the underlying Markov process that is a strong lumping (see Def. 1) [15].

We now introduce the notion of exact equivalence for stochastic automata. An equivalence relation over $\mathcal{S}^*$ is an *exact equivalence* if for any action type $a \in \mathcal{T}^*$, the total conditional transition rates from two equivalence classes to two equivalent states, via activities of this type, are the same. Moreover, for any type a, equivalent states have the same apparent conditional exit rate.

Definition 10. (Exact equivalence) *An equivalence relation $\mathcal{R} \subseteq \mathcal{S}^* \times \mathcal{S}^*$ is an* exact equivalence *if whenever $(s, s') \in \mathcal{R}$ then for all $a \in \mathcal{T}^*$ and for all $C \in \mathcal{S}^*/\mathcal{R}$ it holds*

- *$q(s, a) = q(s', a)$,*
- *$\sum_{s'' \in C} q(s'', s, a) = \sum_{s'' \in C} q(s'', s', a)$.*

The transitive closure of a union of exact equivalences is still an exact equivalence. Hence, the maximal exact equivalence, denoted $\sim_e$, is defined as the union of all exact equivalences. We say that two stochastic automata P and Q are *exactly equivalent*, denoted $P \sim_e Q$, if there exists $s_p \in \mathcal{S}_P$ and $s_q \in \mathcal{S}_Q$ such that $(s_p, s_q) \in \sim_e$.

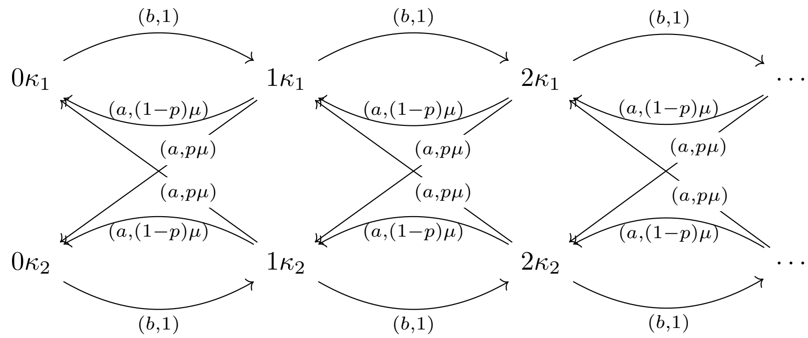

Fig. 2: Queue with alternating servers.

Example 2. Let us consider a queueing model of a system with two identical processors, named κ_1 and κ_2. Each job is assigned to one of the processors which are assumed not to work in parallel. At each service completion event of processor κ_i, the next job is assigned to κ_j, for $i \neq j$, with probability p, and is assigned to processor κ_i with probability $1 - p$. Automaton P underlying this model is depicted in Figure 2 where state $n\kappa_i$, for $n > 0$ and $i = 1, 2$, denotes the state in which processor κ_i is being used and there are n customers waiting to be served. State $0\kappa_i$ denotes the empty queue. It is easy to prove that the equivalence relation $\sim$ obtained by the reflexive closure of $\{(n\kappa_1, n\kappa_2), (n\kappa_2, n\kappa_1) : n \in \mathbb{N}\}$ is an exact equivalence over the state space of P. Let us consider the automaton Q depicted in Figure 1, then it holds that the equivalence relation given by the symmetric and reflexive closure of $\sim' = \sim \cup \{(n\kappa_1, n), (n, n\kappa_2) : n \in \mathbb{N}\}$, where each n denotes a state of Q, is still an exact equivalence. $\square$

The next proposition states that, for any stochastic automaton P, $\sim_e$ induces an exactly lumpable partition on the state space of the Markov process underlying P.

Proposition 4. (Exact lumpability) *Let P be a closed stochastic automaton with state space $\mathcal{S}_P$ and $X_P(t)$ its underlying Markov chain with infinitesimal generator matrix $\mathbf{Q}$. Then for any equivalence class $S_i, S_j \in \mathcal{S}_P/\sim_e$ and $s, s' \in S_i$,*

$$\sum_{s'' \in S_j} q(s'', s) = \sum_{s'' \in S_j} q(s'', s')$$

i.e., $\sim_e$ is an exact lumpability for $X_P(t)$.

The next theorem plays an important role in studying the product-form of exactly equivalent automata. Informally, it states that the exact equivalence preserves the invariant measure of equivalent states.

Theorem 2. *Let P and Q be two closed stochastic automata such that $P \sim_e Q$ and let $\boldsymbol{\mu}_P$ and $\boldsymbol{\mu}_Q$ be two invariant measures of P and Q, respectively. Then,* *there exists a positive constant K such that for each $s_1 \in \mathcal{S}_P$ and $s_2 \in \mathcal{S}_Q$ with $s_1 \sim_e s_2$ it holds that $\mu_P(s_1)/\mu_Q(s_2) = K$.*

Corollary 1. *Let P and Q be two closed stochastic automata such that $P \sim_e Q$ and let $\boldsymbol{\pi}_P$ and $\boldsymbol{\pi}_Q$ be the stationary distributions of P and Q, respectively. Then, for all $s_1, s_2 \in \mathcal{S}_P$ and $s'_1, s'_2 \in \mathcal{S}_Q$ such that $s_i \sim_e s'_i$ for $i = 1, 2$, it holds that $\pi_P(s_1)/\pi_P(s_2) = \pi_Q(s'_1)/\pi_Q(s'_2)$.*

We can prove that both lumpable bisimulation and exact equivalence are congruences for SA synchronisation.

Proposition 5. (Congruence) *Let P, P', Q, Q' be stochastic automata.*

- *If $P \sim_s P'$ and $Q \sim_s Q'$ then $P \otimes Q \sim_s P' \otimes Q'$.*
- *If $P \sim_e P'$ and $Q \sim_e Q'$ then $P \otimes Q \sim_e P' \otimes Q$.*

The proof is based on the fact that if $\sim_i$ is a lumpable bisimulation (resp. an exact equivalence) then the relation

$$\mathcal{R} = \{((s_{p_1}, s_{q_1}), (s_{p_2}, s_{q_2}))|\ s_{p_1} \sim_i s_{p_2} \text{ and } s_{q_1} \sim_i s_{q_2}\}$$

is also a a lumpable bisimulation (resp., an exact equivalence) over $\mathcal{S}_P \times \mathcal{S}_Q$.

The next theorem proves that any exact equivalence between two stochastic automata induces a lumpabale bisimulation between the corresponding reversed automata.

Theorem 3. (Exact equivalence and lumpable bisimulation) *Let P and Q be two closed stochastic automata, P^R and Q^R be the corresponding reversed automata defined according to Definition 6 and $\sim \subseteq \mathcal{S}_P \times \mathcal{S}_Q$ be an exact equivalence. Then $\sim' = \{(s_1^R, s_2^R) \in \mathcal{S}_{P^R} \times S_{Q^R}|\ (s_1, s_2) \in \sim\}$ is a lumpable bisimulation.*

As a consequence any exact equivalence over the state space of a stochastic automaton P induces a lumpable bisimulation over the state space of the reversed automaton P^R.

Corollary 2. *Let P be a closed stochastic automaton and $\sim \subseteq \mathcal{S}_P \times \mathcal{S}_P$ be an exact equivalence. Then the relation $\sim' = \{(s_1^R, s_2^R) \in \mathcal{S}_{P^R} \times \mathcal{S}_{P^R}|\ (s_1, s_2) \in \sim\}$ is a lumpable bisimulation.*

The following lemma provides a characterization of quasi-reversibility in terms of lumpable bisimulation. Informally, it states that an automaton is quasi-reversible if and only if for each closure its reversed is lumpable bisimilar to an automaton with a single state.

Lemma 1. (Quasi-reversibility and lumpable bisimulation) *An irreducible stochastic automaton P is quasi-reversible if and only if the following properties hold for every closure P^c of P with reversed automaton P^{cR}:*

- *if $s^R \in \mathcal{S}_{P^{cR}}$, then $[s^R]_{\sim_s} = \mathcal{S}_{P^{cR}}$*
- *if $a \in \mathcal{P}_P$ then $q_P(s, a) = 1$ for all $s \in \mathcal{S}_P$.*

The following proposition states that both lumpable bisimulations and exact equivalences are invariant with respect to the closure of automata where any closure P^c of P is defined according to Definition 7.

Proposition 6. *Let P and Q be two stochastic automata with $\mathcal{A}_P = \mathcal{A}_Q$, $\mathcal{P}_P = \mathcal{P}_Q = \{a_1, \dots, a_n\}$ and $\sim \subseteq \mathcal{S}_P \times \mathcal{S}_Q$ be an exact equivalence (resp., a lumpable bisimulation). Then for every closure $P^c = P\{a_1 \leftarrow \lambda_1\} \dots \{a_n \leftarrow \lambda_n\}$ and $Q^c = Q\{a_1 \leftarrow \lambda_1\} \dots \{a_n \leftarrow \lambda_n\}$ the relation $\sim' = \{(s_1^c, s_2^c) \in \mathcal{S}_{P^c} \times \mathcal{S}_{Q^c} |\ (s_1, s_2) \in \sim\}$ is an exact equivalence (resp., a lumpable bisimulation).*

The next theorem proves that the class of quasi-reversible stochastic automata is closed under exact equivalence.

Theorem 4. *Let P and Q be two stochastic automata such that $P \sim_e Q$. If Q is quasi-reversible then also P is quasi-reversible.*

Proof. We have to prove that:

1. The outgoing transitions for each passive type $a \in \mathcal{P}_P$ sums to unity.
2. For each closure $P^c = P\{a_1 \leftarrow \lambda_1\} \dots \{a_n \leftarrow \lambda_n\}$ of P with $\lambda_1, \dots, \lambda_n \in \mathbb{R}^+$ there exists a set of positive real numbers $\{k_1, \dots, k_m\}$ such that for each $s \in \mathcal{S}_{P^c}$ and $1 \le i \le m$, Equation (3) is satisfied.

The first claim follows immediately from the first item of Definition 10. Now observe that, by Definition 10, if $P \sim_e Q$ then $\mathcal{P}_P = \mathcal{P}_Q$ and $\mathcal{A}_P = \mathcal{A}_Q$. Let $\mathcal{P}_P = \mathcal{P}_Q = \{a_1, \dots, a_n\}$ and $\mathcal{A}_P = \mathcal{A}_Q = \{b_1, \dots, b_m\}$. By Proposition 6, for any closure $P^c = P\{a_1 \leftarrow \lambda_1\} \dots \{a_n \leftarrow \lambda_n\}$ and $Q^c = Q\{a_1 \leftarrow \lambda_1\} \dots \{a_n \leftarrow \lambda_n\}$ the relation $\sim' = \{(s_1^c, s_2^c) \in \mathcal{S}_{P^c} \times \mathcal{S}_{Q^c} |\ (s_1, s_2) \in \sim$ and $(s_1, s_2) \in \mathcal{S}_P \times \mathcal{S}_Q\}$ is an exact equivalence. By Theorem 3, the relation $\sim'' = \{(s_1^{cR}, s_2^{cR}) \in \mathcal{S}_{P^{cR}} \times S_{Q^{cR}} |\ (s_1, s_2) \in \sim'\}$ is a lumpable bisimulation. By Lemma 1 since Q is quasi-reversible then for all $s^R \in \mathcal{S}_{Q^{cR}}$ it holds $[s^R]_{\sim_s} = \mathcal{S}_{Q^{cR}}$, i.e., there exists a set of positive real numbers $\{k_{b_1}, \dots, k_{b_m}\}$ such that for each $s^R \in \mathcal{S}_{Q^{cR}}$ and $1 \le i \le m$

$$k_{b_i} = \sum_{s' \in \mathcal{S}_{Q^{cR}}} q_{Q^{cR}}(s^R, s', b_i) = \frac{\sum_{s' \in \mathcal{S}_{Q^c}} \mu_{Q^c}(s') q_{Q^c}(s', s, b_i)}{\mu_{Q^c}(s)} ,$$

which can be written as

$$k_{b_i} = \frac{\sum_{C \in \mathcal{S}_{Q^c}/\sim_e} \sum_{s' \in C} \mu_{Q^c}(s') q_{Q^c}(s', s, b_i)}{\mu_{Q^c}(s)} .$$

By Proposition 4, $\sim_e$ induces an exact lumping on the CTMC underlying Q^c and, by Proposition 3, for all s and s' in the same equivalence class $\mu_{Q^c}(s) = \mu_{Q^c}(s')$. Hence we can write

$$k_{b_i} = \frac{\sum_{C \in \mathcal{S}_{Q^c}/\sim_e} \mu_{Q^c}(C) \sum_{s' \in C} q_{Q^c}(s', s, b_i)}{\mu_{Q^c}(s)} .$$

where $\mu_{Q^c}(C)$ denotes $\mu_{Q^c}(s)$ for an arbitrary state $s \in C$. Now from the fact that $P^c \sim_e Q^c$, we have that for each class $C \in \mathcal{S}_{Q^c}/\sim_e$ there exists a class $C' \in \mathcal{S}_{P^c}/\sim_e$ such that all the states $s \in C$ are equivalent to the states in C'. Moreover, by Definition 10, we have $\sum_{s' \in C} q_{Q^c}(s', s_1, b_i) = \sum_{s' \in C'} q_{P^c}(s', s_2, b_i)$ for every state $s_1 \sim_e s_2$ with $s_1 \in Q^c$ and $s_2 \in P^c$. Therefore, we can write:

$$k_{b_i} = \frac{\sum_{C \in \mathcal{S}_{Q^c}/\sim_e} \mu_{Q^c}(C) \sum_{s' \in C} q_{Q^c}(s', s_1, b_i)}{\mu_{Q^c}(s_1)}$$
$$= \frac{\sum_{C \in \mathcal{S}_{Q^c}/\sim_e} \mu_{Q^c}(C) \sum_{s' \in C} q_{P^c}(s', s_2, b_i)}{\mu_{Q^c}(s_1)}$$
$$= \frac{\sum_{C' \in \mathcal{S}_{P^c}/\sim_e} K\mu_{P^c}(C') \sum_{s' \in C'} q_{P^c}(s', s_2, b_i)}{K\mu_{P^c}(s_2)}$$
$$= \frac{\sum_{s' \in \mathcal{S}_{P^c}} \mu_{P^c}(s') q_{P^c}(s', s_2, b_i)}{\mu_{P^c}(s_2)},$$

where K is the positive constant given by Theorem 2. Summing up, since every closure Q^c of Q corresponds to a closure P^c for P and Q^c satisfies Equation (3) for all states s and active types b_i, then the set of positive rates $\{k_{b_i}\}$ defined for Q^c are the same that satisfy Equation (3) for P^c. Therefore, P is also quasi-reversible. □

Example 3. Let us consider the automata Q and P depicted in Figure 1 and Figure 2, respectively. We already observed in Example 2 that there exists an exact equivalence $\sim'$ such that $n, n\kappa_1$ and $n\kappa_2$ belong to the same equivalence class, where n is a state of Q and $n\kappa_i$ belongs to the state space of P. Then, since Q is well-known to be quasi-reversible, by Theorem 2 also P is quasi-reversible. As a consquence, the queueing station modelled by P can be embdded in quasi-reversible product-form queueing networks maintining the property that the equilibrium distribution is separable. □

The next example shows that, differently from exact equivalence, lumpable bisimulation does not preserve quasi-reversibility.

Example 4. Consider the automaton R depicted in Figure 3. It is easy to prove that R is lumpable bisimilar to Jackson's queue Q depicted in Figure 1. However, R is not quasi-reversible, i.e., the corresponding reversed automaton is not lumpable bisimilar to a single-state automaton. More precisely, one can observe that in the reversed automaton there is one type a transition exiting from state 0^R but there is no type a transition from state $1'^R$. This is sufficient to claim that states 0^R and $1'^R$ cannot belong to the same equivalence class. □

The following final result is an immediate consequence of Theorems 1 and 4.

Corollary 3. *Let P, P', Q, Q' be stochastic automata such that $P \sim_e P'$ and $Q \sim_e Q'$. If P and Q have a quasi-reversibility based product-form then also P' and Q' are in product-form.*

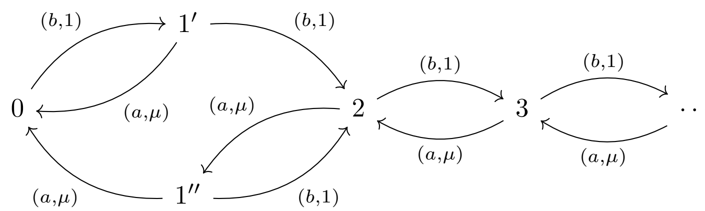

Fig. 3: Stochastic automaton strongly equivalent to a Jackson queue.

6 Conclusion

In this paper we have introduced the notion of exact equivalence, defined on the states of cooperating stochastic automata [24]. With respect to most stochastic equivalences defined for process algebras, exact equivalence induces an exact lumping in the underlying CTMC rather than a strong lumping. We show that this fact has important implications not only from a theoretical point of view but also in reducing the computational complexity of the analysis of cooperating models in equilibrium. Indeed, the class of quasi-reversible automata, whose composition is known to be in product-form and hence analysable efficiently, is closed under the exact equivalence. This leads to a new approach for proving the quasi-reversibility of a stochastic component which does not require to study the reverse-time underlying CTMC but to find a model exactly equivalent to the considered one that is already known to be (or to be not) quasi-reversible, or whose quasi-reversibility can be decided easier.

References

1. S. Baarir, M. Beccuti, C. Dutheillet, G. Franceschinis, and S. Haddad. Lumping partially symmetrical stochastic models. *Perf. Eval.*, 68(1):21 – 44, 2011.
2. C. Baier, J.-P. Katoen, and H. Hermanns. Comparative branching-time semantics for Markov chains. *Inf. Comput.*, 200(2):149–214, 2005.
3. S. Balsamo and A. Marin. *Queueing Networks in Formal methods for performance evaluation*, chapter 2, pages 34–82. M. Bernardo and J. Hillston (Eds), LNCS, Springer, 2007.
4. S. Balsamo, G. Dei Rossi, and A. Marin. Lumping and reversed processes in cooperating automata. *Annals of Oper. Res.*, 2014.
5. F. Baskett, K. M. Chandy, R. R. Muntz, and F. G. Palacios. Open, closed, and mixed networks of queues with different classes of customers. *J. ACM*, 22(2):248–260, 1975.
6. M. Bernardo. Weak Markovian bisimulation congruences and exact CTMC-level aggregations for concurrent processes. In *Proc. of the 10th Workshop on Quantitative Aspects of Programming Languages and Systems (QALP12)*, pages 122–136, 2012.
7. M. Bernardo and R. Gorrieri. A tutorial on EMPA: A theory of concurrent processes with nondeterminism, priorities, probabilities and time. *Theoretical Computer Science*, 202:1–54, 1998.

8. M. Bravetti. Revisiting interactive Markov chains. *Electr. Notes Theor. Comput. Sci.*, 68(5):65–84, 2003.
9. P. Buchholz. Exact and ordinary lumpability in finite Markov chains. *Journal of Applied Probability*, 31:59–75, 1994.
10. P. Buchholz. Exact performance equivalence: An equivalence relation for stochastic automata. *Theor. Comput. Sci.*, 215(1-2):263–287, 1999.
11. P. G. Harrison. Turning back time in Markovian process algebra. *Theoretical Computer Science*, 290(3):1947–1986, 2003.
12. H. Hermanns. *Interactive Markov Chains.* Springer, 2002.
13. H. Hermanns, U. Herzog, and J. P. Katoen. Process algebra for performance evaluation. *Theor. Comput. Sci.*, 274(1-2):43–87, 2002.
14. J. Hillston. *A Compositional Approach to Performance Modelling.* Cambridge Press, 1996.
15. J. Hillston, A. Marin, C. Piazza, and S. Rossi. Contextual lumpability. In *Proc. of Valuetools 2013 Conf.* ACM Press, 2013.
16. A. Skou K. G. Larsen. Bisimulation through probabilistic testing. *Inf. Comput.*, 94(1):1–28, 1991.
17. F. Kelly. *Reversibility and stochastic networks.* Wiley, New York, 1979.
18. J. G. Kemeny and J. L. Snell. *Finite Markov Chains.* Springer, 1976.
19. J. Y. Le Boudec. A BCMP extension to multiserver stations with concurrent classes of customers. In *SIGMETRICS '86/PERFORMANCE '86: Proc. of the 1986 ACM SIGMETRICS Int. Conf. on Computer performance modelling, measurement and evaluation*, pages 78–91, New York, NY, 1986. ACM Press.
20. A. Marin and S. Rossi. Autoreversibility: exploiting symmetries in Markov chains. In *Proc. of MASCOTS 2013*, pages 151–160. IEEE Computer Society, 2013.
21. A. Marin and M. G. Vigliotti. A general result for deriving product-form solutions of Markovian models. In *Proc. of First Joint WOSP/SIPEW Int. Conf. on Perf. Eng.*, pages 165–176, San Josè, CA, USA, 2010. ACM.
22. M. K. Molloy. Performance analysis using stochastic Petri nets. *IEEE Trans. on Comput.*, 31(9):913–917, 1982.
23. R. Paige and R. E. Tarjan. Three Partition Refinement Algorithms. *SIAM Journal on Computing*, 16(6):973–989, 1987.
24. B. Plateau. On the stochastic structure of parallelism and synchronization models for distributed algorithms. *SIGMETRICS Perf. Eval. Rev.*, 13(2):147–154, 1985.
25. P. Schweitzer. Aggregation methods for large Markov chains. *Mathematical Computer Performance and Reliability*, 1984.
26. J. Sproston and S. Donatelli. Backward bisimulation in Markov chain model checking. *IEEE TSE*, 32(8):531–546, 2006.
27. U. Sumita and M. Reiders. Lumpability and time-reversibility in the aggregation-disaggregation method for large Markov chains. *Communications in Statistics - Stochastic Models*, 5:63–81, 1989.