Feasible and Infeasible Maxima in a Quadratic Program for Maximum Clique

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In 1965, Motzkin and Straus related global maxima of a certain quadratic program to the maximum clique size in a certain graph. We extend this result to relate strict local maxima of this program to certain maximal cliques, and certain maxima to non cliques. Our results are useful to a companion paper which employs this QP in a neural net model to find large cliques in graphs.

1 INTRODUCTION

Let $G = (V, E)$ be an undirected graph, where $V = \{1, \cdots, n\}$ is the set of vertices, and $E \subseteq V \times V$ is the set of edges. Vertices i and j are called *adjacent* if they are connected by an edge. A *clique* of G is a subset of V in which every pair of vertices is adjacent. A clique C is called *maximal* if no strict superset of C is a clique. The highest-cardinality maximal clique is called a *maximum* clique. The *maximum clique problem* is to find a maximum clique in a given graph G. The problem is NP-hard [1], even to approximate well [2].

[0]The authors thank J. Shawe-Taylor for an interesting observation about Proposition 7, which led to Corollary 9. A.J. thanks K.W. Regan who first pointed out, when shown [5], infeasible global maxima in the 3-vertex graph with two edges. A.J. thanks A. Baveja for joint exploratory experiments and conjectures with Mathematica. Baveja conjectured that if two maximum cliques of cardinality k intersect in $k-1$ vertices, then every convex combination of their characteristic vectors is a global maximum. Eventually this lead A.J. to discover Theorem 6 and its proof.

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Let $A_g = (a_{ij})$ denote the *adjacency matrix* of G: $a_{ij} = 1$ if $i \neq j$ and vertices i and j are adjacent, $a_{ij} = 0$ otherwise. For a given set $S \subseteq V$, define its *characteristic vector* $\mathbf{x}^s$ as:

$$x_i^s = \begin{cases} 1/|S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

where $|S|$ denotes the cardinality of S. Given two disjoint sets S and T of vertices, we say that a set E' of edges *crosses* S and T if every edge in E' has one endpoint in S and the other in T. The set E' is called a *join* if $|E'| = |S| \times |T|$. The set E' is called a *matching* if no two edges in E' have the same endpoint. Given two sets A and B, $A \backslash B$ is the set of elements in A but not in B. Finally, $\binom{n}{k} = n!/(k!(n-k)!)$ is the number of k-element subsets of an n-element set.

Given a graph G, consider the following quadratic program [3, 4]:

$$\begin{array}{ll} \text{maximize} & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A_g \mathbf{x} \\ \text{subject to} & \mathbf{x} \in \mathcal{K} \end{array} \tag{1}$$

where $\mathcal{K} = \{\mathbf{x} \in \Re^n \;:\; x_i \geq 0, \text{ for all } i = 1 \ldots n, \text{ and } \sum_{i=1}^n x_i = 1\}$.

A point $\mathbf{x}^*$ is said to be a *global maximum* of (1) if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{K}$. It is said to be a *local maximum* if there exists an $\epsilon > 0$ such that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{K}$ satisfying $||\mathbf{x} - \mathbf{x}^*|| < \epsilon$ and a *strictly local maximum* if $f(\mathbf{x}^*) > f(\mathbf{x})$ for $\mathbf{x}$ chosen as above.

Motzkin and Straus [3] proved the following theorem which relates a global maximum of (1) to the maximum clique size of G (see also [4] for a proof).

Theorem 1 *Let $\mathbf{x}^*$ be a global maximum of (1). Then, for every maximum clique C in G, $f(\mathbf{x}^c) = f(\mathbf{x}^*) = \alpha$ and $|C| = 1/(1-2\alpha)$.*

In a companion paper [6], we exploit this theorem to map the maximum clique problem onto a simple and effective parallel relaxation network. In this note, we extend the result of Motzkin and Straus to gain more insight into the connection between maxima of (1) and cliques in graphs. We characterize strictly local maxima as certain kinds of maximal cliques. We show the existence of infinitely-many infeasible maxima, i.e. ones that are not characteristic vectors of cliques, and characterize them. These results provide insight into the operation of optimization procedures for (1), in particular on that of the network of [6], and on the optimization methods in [4].

2 FEASIBLE MAXIMA

Pardalos and Phillips [4] showed that not all global maxima of (1) correspond to maximum cliques. Our results in Section 3 point to the same conclusion. In this section, we show that if a global maximum is a characteristic vector of a subset S of vertices, then S is indeed a maximum clique.

Proposition 2 *A subset C of vertices in G is a maximum clique if and only if its characteristic vector $\mathbf{x}^c$ is a global maximum of (1).*

Proof : One direction is immediate from Theorem 1. For the other direction, suppose $\mathbf{x}^c$ is a global maximum of (1). From Theorem 1, $f(\mathbf{x}^c) = f(\mathbf{x}^s)$ where S is a maximum clique in G. Let $|C| = c$ and $|S| = s$. From (1), $f(\mathbf{x}^c) \leq \binom{c}{2}/c^2$ and $f(\mathbf{x}^s) = \binom{s}{2}/s^2$. But $f(\mathbf{x}^c) = f(\mathbf{x}^s)$ only if $f(\mathbf{x}^c) = \binom{c}{2}/c^2 = \binom{s}{2}/s^2$ which implies that $c = s$ and, from (1), that C is a clique. Hence C is a maximum clique. ∎

A point $\mathbf{x}$ satisfies the Kuhn-Tucker conditions [7] for problem (1) if there exists a real constant λ and n nonnegative constants $\mu_1, \cdots, \mu_n$ such that:

$$\sum_{j=1}^{n} a_{ij} x_j - \lambda + \mu_i \quad = \quad 0 \text{ for all } i = 1 \ldots n; \text{ and } \sum_{i=1}^{n} \mu_i x_i = 0.$$

Note that, since $\mu_i \geq 0$ and $x_i \geq 0$, the latter condition is equivalent to saying that $x_i > 0$ implies $\mu_i = 0$ for all $i = 1 \ldots n$. We may therefore write the Kuhn-Tucker conditions for this problem as

$$\sum_{j=1}^{n} a_{ij} x_j \begin{cases} = & \lambda \quad \text{if } x_i > 0 \\ \leq & \lambda \quad \text{if } x_i = 0 \end{cases} \tag{2}$$

for some real constant λ. A useful characterization of maximal cliques is as follows.

Proposition 3 *Let C be a subset of k vertices of a graph G. Then C is a maximal clique of G if and only if its characteristic vector $\mathbf{x}^c$ satisfies*

$$\sum_{j=1}^{n} a_{ij} x_j^c \begin{cases} = & \frac{k-1}{k} \quad \textit{if } i \in C \\ \leq & \frac{k-1}{k} \quad \textit{if } i \notin C \end{cases} \tag{3}$$

Proof : Suppose C is a maximal clique. From the definition of maximal clique $\sum_{j \in C} a_{ij} = k - 1$ for all $i \in C$ and $\sum_{j \in C} a_{ij} \leq k - 1$ for all $i \notin C$. Now observing that

$$\sum_{j=1}^{n} a_{ij} x_j^c = \frac{1}{k} \sum_{j \in C} a_{ij} \text{ for all } i = 1 \ldots n.$$

we conclude that $\mathbf{x}^c$ satisfies (3). The other direction is equally simple. ∎

The following corollary follows immediately, which establishes a first connection between the local maxima of problem (1) and the maximal cliques of G.

Corollary 4 *If C is a maximal clique, $\mathbf{x}^c$ satisfies the Kuhn-Tucker conditions for program (1).*

Definition 1 *A maximal clique C is said to be* strictly *maximal if, for all $i \in V \backslash C$, the number of edges crossing $\{i\}$ and C is less than $|C|-1$.*

Note that for C to be a maximal clique, it suffices that the number of edges crossing $\{i\}$ and C be no more than $|C|-1$. Definition 1 requires this number to be strictly less than $|C|-1$. We are now in a position to prove the main result of this section.

Theorem 5 *Let $G=(V,E)$ be a graph and let C be a subset of V. Then C is a strictly maximal clique if and only if $\mathbf{x}^c$ is a strict local maximum of (1).*

Proof : Suppose that $\mathbf{x}^c$ is a strict local maximum of (1). Then the Kuhn-Tucker conditions (2) hold for some λ. We will show that $\lambda = \frac{k-1}{k}$ where $k=|C|$ from which, by Proposition 3, it will follow that C is a maximal clique. Suppose, to the contrary, that this is not the case, and let i and j be two vertices in C that are not adjacent, i.e., $a_{ij}=a_{ji}=0$. Let $\epsilon \leq 1/k$ be an arbitrarily small positive constant, and construct the vector $\mathbf{z}_\epsilon$ as follows: $\mathbf{z}_\epsilon = \mathbf{x}^c + \epsilon(\mathbf{e}_i - \mathbf{e}_j)$, where $\mathbf{e}_l$ denotes the l-th standard base vector of $\Re^n$, i.e., $e_{lh}=1$ if $l=h$ and $e_{lh}=0$ otherwise. Note that $\mathbf{z}_\epsilon$ is a feasible point for the domain of program (1). We have:

$$
\begin{aligned}
f(\mathbf{z}_\epsilon) &= \frac{1}{2}{\mathbf{z}_\epsilon}^T A_g \mathbf{z}_\epsilon = \frac{1}{2}\left(\mathbf{x}^c + \epsilon(\mathbf{e}_i - \mathbf{e}_j)\right)^T A_g \left(\mathbf{x}^c + \epsilon(\mathbf{e}_i - \mathbf{e}_j)\right) \\
&= \frac{1}{2}\left\{\mathbf{x}^{cT} A_g \mathbf{x}^c + 2\epsilon(\mathbf{e}_i - \mathbf{e}_j)^T A_g \mathbf{x}^c + \epsilon^2(\mathbf{e}_i - \mathbf{e}_j)^T A_g (\mathbf{e}_i - \mathbf{e}_j)\right\} .
\end{aligned}
$$

Now, observe that from (2)

$$
(\mathbf{e}_i - \mathbf{e}_j)^T A_g \mathbf{x}^c = \sum_{l=1}^n a_{il} x_l^c - \sum_{l=1}^n a_{jl} x_l^c = \lambda - \lambda = 0
$$

and, since $a_{ij}=a_{ji}=0$, $(\mathbf{e}_i - \mathbf{e}_j)^T A_g (\mathbf{e}_i - \mathbf{e}_j) = 0$ so that $f(\mathbf{z}_\epsilon) = f(\mathbf{x}^c)$, for all positive $\epsilon \leq 1/k$. This contradicts the hypothesis that $\mathbf{x}^c$ is a strict local maximum of f, thereby proving that C is a maximal clique.

To see that C is a *strictly* maximal clique, suppose to the contrary that $j \notin C$ is adjacent to exactly $k-1$ vertices in C, and let i denote the (only) vertex in C not adjacent to j. The set $D = \{j\} \cup (C \backslash \{i\})$ is a clique of the same cardinality as C. The conditions to apply Theorem 6 of the next section are met, because $m = |C \backslash D| = |D \backslash C| = 1$, and there are no edges crossing $C \backslash D$ and $D \backslash C$, since i and j are non-adjacent. From Theorem 6, for all $\alpha \in [0,1]$ we have $f(\mathbf{x}^c) = f(\alpha \mathbf{x}^c + (1-\alpha)\mathbf{x}^d)$ which contradicts the hypothesis that $\mathbf{x}^c$ is a *strict* local maximum of (1). This proves the first part of the theorem.

For the other part, suppose that C is a strictly maximal clique. To prove that $\mathbf{x}^c$ is a strict local maximum of program (1) we apply the second-order sufficiency conditions for constrained optimization [7]. First, from Corollary 4, $\mathbf{x}^c$ satisfies the Kuhn-Tucker conditions. Note that in this case, the Lagrange multipliers μ_i's are given by

$$\mu_i = \frac{k-1}{k} - \sum_{j=1}^{n} a_{ij} x_j^c \ .$$

It remains to be shown that the Hessian of the Lagrangian associated with program (1), which in our case is simply A_g, is negative definite on the subspace

$$M = \left\{ \mathbf{y} \in \Re^n \ : \ \sum_{i=1}^{n} y_i = 0, \ \text{ and } \ y_i = 0 \ \text{ for all} i \in I \right\}$$

where $I = \{i \ : \ x_i^c = 0 \ \text{ and } \ \mu_i > 0\}$. Since C is a strictly maximal clique, we have

$$\sum_{j=1}^{n} a_{ij} x_j^c < \frac{k-1}{k} \ , \quad \text{for all } i \notin C$$

so that $I = V \backslash C$. Now, let $\mathbf{y} \in M \backslash \{\mathbf{0}\}$:

$$\mathbf{y}^T A_g \mathbf{y} \ = \ \sum_{i=1}^{n} y_i \sum_{j=1}^{n} a_{ij} y_j = \sum_{i \in C} y_i \sum_{j \in C} a_{ij} y_j = \sum_{i \in C} y_i \left(\sum_{j \in C} y_j - y_i \right) = - \sum_{i \in C} y_i^2 < 0$$

which proves the theorem. ■

Gibbons *et al.* [8] have independently proven the preceding theorem, and also a number of other interesting results concerning the Motzkin-Straus program.

3 INFEASIBLE MAXIMA

We say that a local maximum $\mathbf{x}$ of program (1) is *infeasible* if $\mathbf{x}$ is not a characteristic vector of some maximal clique in the associated graph. A maximum $\mathbf{x}$ might be infeasible because it is not a characteristic vector, or because it is a characteristic vector of some subset S of vertices that does not form a maximal clique. The latter possibility is almost ruled out by Theorem 5. It would be fully ruled out if it turned out that Theorem 5 could be strengthened to remove the qualifier "strict" from its statement, a question that is currently open.

In this section, we characterize infeasible global maxima (1) that are not characteristic vectors of some subset of vertices. Our characterization sheds light on the possibility that an optimization procedure, for example the one in [6], may retrieve such global maxima during the course of its operation. Our characterization also suggests how, in some cases, one might extract a clique from such a retrieved maximum, as explained later.

The existence of infeasible maxima was earlier observed by Pardalos and Phillips [4]. Here we characterize these maxima when the graph has a certain structure, and show that there are infinitely-many infeasible maxima in such a case.

Theorem 6 *Consider a graph G which contains two cliques C and D of equal cardinality $|C| = |D| = k$. Let $|C \backslash D| = |D \backslash C| = m \leq k$. Then, for every $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$:*

1. *If G has exactly $m(m-1)$ edges crossing $C \backslash D$ and $D \backslash C$ then $f(\alpha_1 \mathbf{x}^c + \alpha_2 \mathbf{x}^d) = f(\mathbf{x}^c)$.*
2. *If G has fewer than $m(m-1)$ edges crossing $C \backslash D$ and $D \backslash C$ then $f(\alpha_1 \mathbf{x}^c + \alpha_2 \mathbf{x}^d) < f(\mathbf{x}^c)$.*

Proof: Note that $|C \cap D| = k - m \geq 0$. Label the vertices $C \cup D$ as $\{1, \ldots, k+m\}$ with $C = \{1, \ldots, k\}$. Then

$$\begin{aligned}
\mathbf{x}^c &= (\underbrace{1/k, \ldots, 1/k}_{m}, \underbrace{1/k, \ldots, 1/k}_{k-m}, \underbrace{0, \ldots, 0}_{m}, 0, \ldots, 0) \\
\mathbf{x}^d &= (\underbrace{0, \ldots, 0}_{m}, \underbrace{1/k, \ldots, 1/k}_{k-m}, \underbrace{1/k, \ldots, 1/k}_{m}, 0, \ldots, 0)
\end{aligned}$$

with $f(\mathbf{x}^c) = f(\mathbf{x}^d) = \binom{k}{2}/k^2$. For every $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$:

$$\begin{aligned}
f(\alpha_1 \mathbf{x}^c + \alpha_2 \mathbf{x}^d) &= f(\underbrace{\alpha_1/k, \ldots, \alpha_1/k}_{m}, \underbrace{1/k, \ldots, 1/k}_{k-m}, \underbrace{\alpha_2/k, \ldots, \alpha_2/k}_{m}, 0, \ldots, 0) \\
&= \binom{m}{2}\alpha_1^2/k^2 + \binom{k-m}{2}/k^2 + \binom{m}{2}\alpha_2^2/k^2 + \\
&\quad m(k-m)\alpha_1/k^2 + m(k-m)\alpha_2/k^2 + e\alpha_1\alpha_2/k^2
\end{aligned} \tag{4}$$

where e is the number of edges crossing $C \backslash D$ and $D \backslash C$. Substituting $e = m(m-1)$ in (4) and simplifying, we get:

$$\begin{aligned}
f(\alpha_1 \mathbf{x}^c + \alpha_2 \mathbf{x}^d) &= \frac{\binom{m}{2}[\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2] + \binom{k-m}{2} + m(k-m)}{k^2} \\
&= \frac{\binom{m}{2} + \binom{k-m}{2} + m(k-m)}{k^2} = \frac{\binom{k}{2}}{k^2} = f(\mathbf{x}^c)
\end{aligned}$$

Obviously if $e < m(m-1)$ then $f(\alpha_1 \mathbf{x}^c + \alpha_2 \mathbf{x}^d) < f(\mathbf{x}^c)$. ∎

Let us now examine the hypotheses of Theorem 6 in more detail.

Proposition 7 *If C and D are maximal cliques in G, then the number of edges crossing $C \backslash D$ and $D \backslash C$ in H cannot exceed $m(m-1)$. This bound is tight. The graphs with exactly $m(m-1)$ edges crossing $C \backslash D$ and $D \backslash C$, with C and D as maximal cliques, are exactly those in which the set of crossing edges is a join minus a perfect matching.*

Proof : Suppose, on the contrary, that there are more than $m(m-1)$ crossing edges. Then there is a vertex i in $C \backslash D$ or in $D \backslash C$ adjacent to every vertex in the other set. If $i \in C$ then $\{i\} \cup D$ is a clique, else $\{i\} \cup C$ is a clique. In either case, the assumption that C and D are maximal cliques in G is violated.

Clearly if a graph has two equal-cardinality maximal cliques C and D, and the edges crossing $C \backslash D$ and $D \backslash C$ are their join minus a perfect matching, the number of crossing edges is $m(m-1)$. Consider any graph in which C and D are cliques, the number of edges crossing $C \backslash D$ and $D \backslash C$ equal $m(m-1)$, and the set of crossing edges does not form a join minus a perfect matching. In this case, some vertex in $C \backslash D$ or in $D \backslash C$ must be adjacent to every vertex in the other set. Thus C or D is not a maximal clique. ■

The following corollary is immediate from Theorem 6 and Proposition 7.

Corollary 8 *Consider a graph G containing maximum cliques C and D (among possibly others). Then every convex combination of their characteristic vectors $\mathbf{x}^c$ and $\mathbf{x}^d$ is a global maximum of (1) if and only if the number of edges crossing $C \backslash D$ and $D \backslash C$ is $m(m-1)$, where $m = |C \backslash D|$.*

J. Shawe-Taylor pointed out that, in the construction of Proposition 7 when the number of crossing edges is exactly $m(m-1)$, there is a boolean lattice of cliques between C and D if one defines the ordering by the amount of intersection with C. Each step in this lattice corresponds to exchanging one pair of nonadjacent vertices from the deleted perfect matching. Let $\mathcal{C}$ denote the set of cliques in this lattice. Note that $|\mathcal{C}|$ equals 2^m.

This leads to the following result.

Corollary 9 *When C and D are maximum cliques in G, for every pair of elements E, F in their lattice $\mathcal{C}$, every convex combination of their characteristic vectors $\mathbf{x}^e$ and $\mathbf{x}^f$ is a global maximum of (1).*

Proof : E and F are clearly maximum cliques in G. Let $k = |E \backslash F|$. Since the edge-set of $C \cup D$ misses only a matching, the number of edges crossing E and F equals $k(k-1)$, hence Corollary 8 is applicable. ■

Interestingly, when C and D are maximal cliques in G, not all cliques in their lattice $\mathcal{C}$ need be maximal cliques in G. Counter-examples are easy to construct.

From an infeasible maximum $\mathbf{x}^*$ which is a convex combination of characteristic vectors of maximum cliques C and D, a maximum clique may be extracted as follows. The largest-valued components constitute $C \cap D$. The perfect matching in the remainder of $C \cup D$ is easy to find; adding all vertices on side of the matching to $C \cap D$ constitutes a maximum clique.

Finally, Theorem 6 can be generalized as follows:

Theorem 10 *Let $\mathbf{x}^1, \cdots, \mathbf{x}^q$ represent characteristic vectors of cliques $C_1, \cdots, C_q$ of equal cardinality $|C_1| = \cdots = |C_q| = k$ of any graph G. Let $|C_h \backslash C_l| = |C_l \backslash C_h| = m_{hl} \leq k$, $(h, l = 1 \ldots q)$. Then, for every $\alpha_1, \cdots, \alpha_q \geq 0$ such that $\sum_{h=1}^{q} \alpha_h = 1$:*

1. *If G has exactly $m_{hl}(m_{hl} - 1)$ edges crossing $C_h \backslash C_l$ and $C_l \backslash C_h$, for all $h, l = 1 \ldots q$, then $f(\alpha_1 \mathbf{x}^1 + \cdots + \alpha_q \mathbf{x}^q) = f(\mathbf{x}^1) = \cdots = f(\mathbf{x}^q)$.*
2. *If G has fewer than $m_{hl}(m_{hl} - 1)$ edges crossing $C_h \backslash C_l$ and $C_l \backslash C_h$, for some $h, l = 1 \ldots q$, then $f(\alpha_1 \mathbf{x}^1 + \cdots + \alpha_q \mathbf{x}^q) < f(\mathbf{x}^1) = \cdots = f(\mathbf{x}^q)$*

Proof: Let $\alpha_1, \cdots, \alpha_q \geq 0$ subject to $\sum_{h=1}^{q} \alpha_h = 1$, and let e_{hl} denote the number of edges crossing $C_h \backslash C_l$ and $C_l \backslash C_h$ $(h, l = 1 \ldots q)$. We have:

$$f\left(\sum_{h=1}^{q} \alpha_h \mathbf{x}^h\right) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \left(\sum_{h=1}^{q} \alpha_h x_i^h\right) \left(\sum_{l=1}^{q} \alpha_l x_j^l\right) = \frac{1}{2k^2} \sum_{h,l=1}^{q} \alpha_h \alpha_l \sum_{i \in C_h,\ j \in C_l} a_{ij} \,.$$

For every $h, l = 1 \ldots q$ we can write:

$$\sum_{i \in C_h,\ j \in C_l} a_{ij} = \sum_{i \in C_h \backslash C_l,\ j \in C_l \backslash C_h} a_{ij} + \sum_{i \in C_h \backslash C_l,\ j \in C_l \cap C_h} a_{ij} + \sum_{i \in C_h \cap C_l,\ j \in C_l \backslash C_h} a_{ij} + \sum_{i,j \in C_h \cap C_l} a_{ij} \,.$$

Note that the first term in the right-hand side of the above equation is equal to e_{hl}. Also, since both C_h and C_l are cliques, we get

$$\sum_{i \in C_h \backslash C_l,\ j \in C_l \cap C_h} a_{ij} = \sum_{i \in C_h \cap C_l,\ j \in C_l \backslash C_h} a_{ij} = m_{hl}(k - m_{hl}),$$

and

$$\sum_{i,j \in C_h \cap C_l} a_{ij} = (k - m_{hl})(k - m_{hl} - 1).$$

Now, if $e_{hl} = m_{hl}(m_{hl} - 1)$ we get, after some simplifications

$$\sum_{i \in C_h} \sum_{j \in C_l} a_{ij} = k(k-1)$$

and thus

$$f\left(\sum_{h=1}^{q} \alpha_h \mathbf{x}^h\right) = \frac{\binom{k}{2}}{k^2} \sum_{h=1}^{q} \alpha_h \sum_{l=1}^{q} \alpha_l = \frac{\binom{k}{2}}{k^2} = f(\mathbf{x}^1) = \cdots = f(\mathbf{x}^q)$$

which proves the first part of the theorem. If $e_{hl} < m_{hl}(m_{hl}-1)$ we readily obtain

$$f(\alpha_1 \mathbf{x}^1 + \cdots + \alpha_q \mathbf{x}^q) < f(\mathbf{x}^1) = \cdots = f(\mathbf{x}^q),$$

and this completes the proof. ∎

Corollary 11 *Consider a graph G containing maximum cliques $C_1, \cdots, C_q$ (amongst possibly others). Then every vector belonging to the convex hull of their characteristic vectors is a global maximum of program (1) if and only if, for all $h, l = 1 \ldots q$, the number of edges crossing $C_h \backslash C_l$ and $C_l \backslash C_h$ equals $m_{hl}(m_{hl}-1)$.*

4 OPEN QUESTIONS

Can Corollary 8 be extended to hold when C and D are maximal cliques rather than maximum cliques and the term "global maximum" replaced by "local maximum"? Can the qualifier "strictly" be removed from Theorem 5?

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