Hopfield Model – Continuous Case

The Hopfield model can be generalized using continuous activation functions.

More plausible model.

In this case:

$$V_i = g_b(u_i) = g_b\left(\sum_j W_{ij} V_j + I_i\right)$$

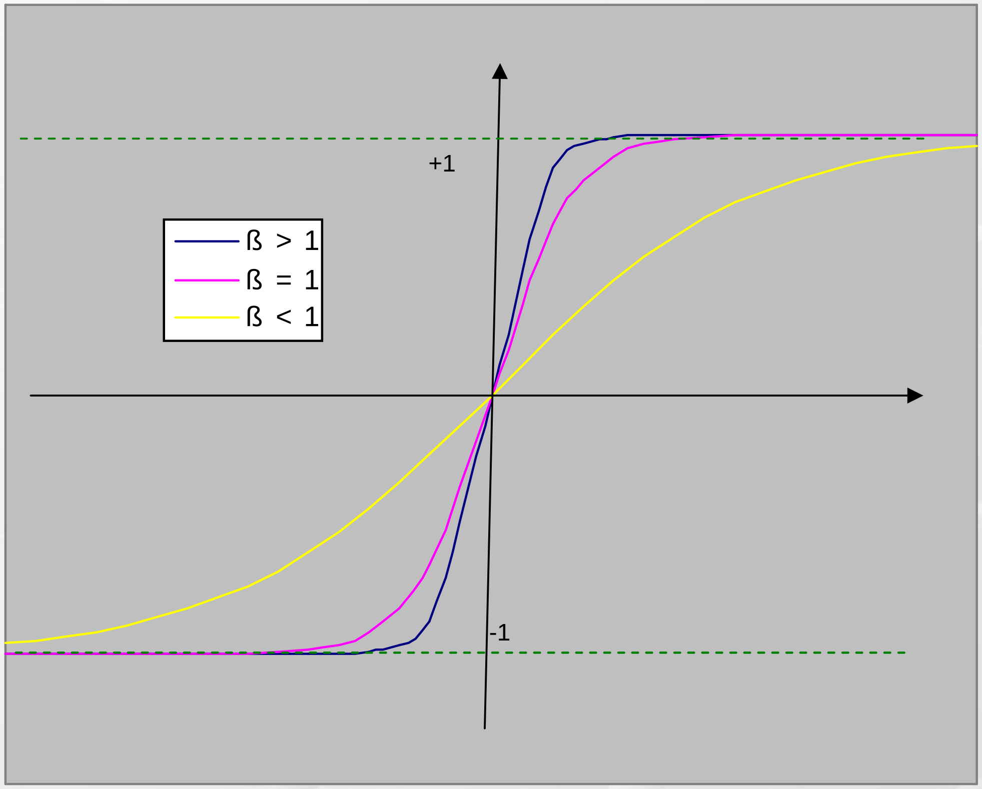
where g_b is a continuous, increasing, non linear function.

Examples

$$tanh(b\,u) = \frac{e^{b\,u} - e^{-b\,u}}{e^{b\,u} + e^{-b\,u}} \qquad \in \left]-1,1\right[$$

$$g_b(u) = \frac{1}{1 + e^{-2\,b\,u}} \qquad \in \left]0,1\right[$$

Funzione di attivazione



$$f(x) = tanh(b\ x)$$

Updating Rules

Several possible choices for updating the units :

Asynchronous updating: one unit at a time is selected to have its output set

Synchronous updating: at each time step all units have their output set

Continuous updating: all units continuously and simultaneously change their outputs

Continuous Hopfield Models

Using the continuous updating rule, the network evolves according to the following set of (coupled) differential equations:

$$t_i \frac{dV_i}{dt} = -V_i + g_b\,(u_i) = -V_i + g_b\left(\sum_j w_{ij} V_j + I_i\right)$$

where t_i are suitable time constants ($t_i > 0$).

Note When the system reaches a fixed point ($dV_i / dt = 0 \quad \forall\, i$) we get

$$V_i = g_b\,(u_i)$$

Indeed, we study a very similar dynamics

$$t_i \frac{du_i}{dt} = -u_i + \sum_j w_{ij}\, g_b\,(u_j) + I_i$$

Modello di Hopfield continuo (energia)

$$\frac{dE}{dt} = -\frac{1}{2}\sum_{ij} T_{ij}\frac{dV_i}{dt}V_j - \frac{1}{2}\sum_{ij} T_{ij}V_i\frac{dV_j}{dt} + \sum_i g_b^{-1}(V_i)\frac{dV_i}{dt} - \sum_i I_i\frac{dV_i}{dt}$$

$$= -\sum_i \frac{dV_i}{dt}\left(\sum_j T_{ij}V_j - u_i + I_i\right)$$

$$= -\sum_i t_i \frac{dV_i}{dt}\frac{du_i}{dt}$$

$$= -\sum_i t_i\, g_b'(u_i)\left(\frac{du_i}{dt}\right)^2 \leq 0$$

Perché g_b è monotona crescente e $t_i > 0$.

N.B. $\frac{dE}{dt} = 0 \quad \Leftrightarrow \quad \frac{du_i}{dt} = 0$

cioè u_i è un punto di equilibrio

The Energy Function

As the discrete model, the continuous Hopfield network has an "energy" function, provided that $W = W^T$:

$$E = -\frac{1}{2}\sum_i \sum_j w_{ij} V_i V_j + \sum_i \int_0^{V_i} g_b^{-1}(V)\, dV - \sum_i I_i V_i$$

Easy to prove that

$$\frac{dE}{dt} \leq 0$$

with equality iff the net reaches a fixed point.

Modello di Hopfield continuo
(relazione con il modello discreto)

Esiste una relazione stretta tra il modello continuo e quello discreto.

Si noti che :

$$V_i = g_b(u_i) = g_1(b\, u_i) \equiv g(b\, u_i)$$

quindi :

$$u_i = \frac{1}{b} g^{-1}(V_i)$$

Il 2º termine in E diventa :

$$\frac{1}{b} \sum_i \int_0^{V_i} g_i^{-1}(V)\, dV$$

L'integrale è positivo (0 se V_i=0).

Per $b \to \infty$ il termine diventa trascurabile, quindi la funzione E del modello continuo diventa identica a quello del modello discreto

Optimization Using Hopfield Network

- Energy function of Hopfield network

$$E = -\frac{1}{2}\sum_i \sum_j w_{ij} V_i V_j - \sum_i I_i V_i$$

- The network will evolve into a (locally / globally) minimum energy state

- Any quadratic cost function can be rewritten as the Hopfield network Energy function. Therefore, it can be minimized using Hopfield network.

- Classical Traveling Salesperson Problem (TSP)

- Many other applications
    - 2-D, 3-D object recognition
    - Image restoration
    - Stereo matching
    - Computing optical flow