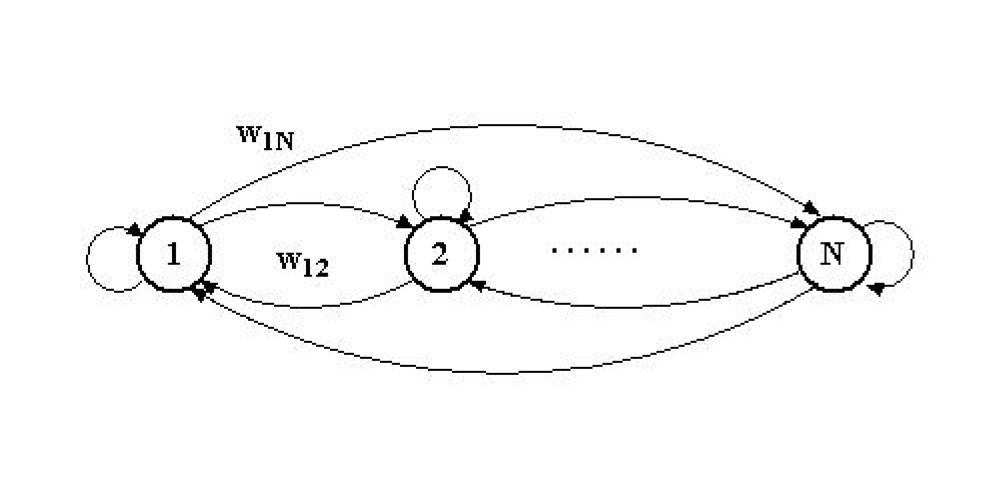
Hopfield Network

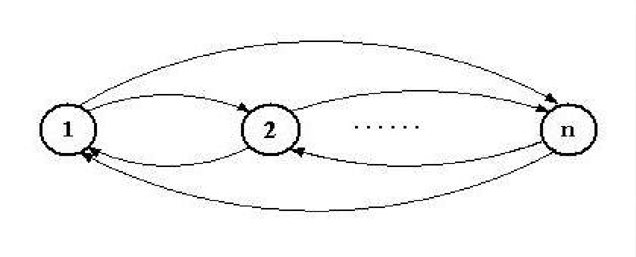
- Single Layer Recurrent Network
- Bidirectional Symmetric Connection
- Binary / Continuous Units
- Associative Memory
- Optimization Problem



Hopfield Model – Discrete Case

Recurrent neural network that uses McCulloch and Pitt's (binary) neurons.

Update rule is stochastic.



Eeach neuron has two "states" : V_i^L , V_i^H

Usually : $\begin{cases} V_i^L = -1 \,,\; V_i^H = 1 \\ V_i^L = 0 \,,\; V_i^H = 1 \end{cases}$

Input to neuron *i* is :

$$H_i = \sum_{j \neq i} w_{ij} V_j + I_i$$

Where:

- w_{ij} = strength of the connection from *j* to *i*
- V_j = state (or output) of neuron *j*
- I_i = external input to neuron *i*

Hopfield Model – Discrete Case

Each neuron updates its state in an *asynchronous* way, using the following rule:

$$V_i = \begin{cases} -1 & if \quad H_i = \sum_{j \neq i} w_{ij} V_j + I_i \; < 0 \\ +1 & if \quad H_i = \sum_{j \neq i} w_{ij} V_j + I_i \; > 0 \end{cases}$$

The updating of states is a *stochastic* process:

To select the to-be-updated neurons we can proceed in either of two ways:

- At each time step select at random a unit *i* to be updated
  (useful for simulation)

- Let each unit independently choose to update itself with
  some constant probability per unit time
  (useful for modeling and hardware implementation)

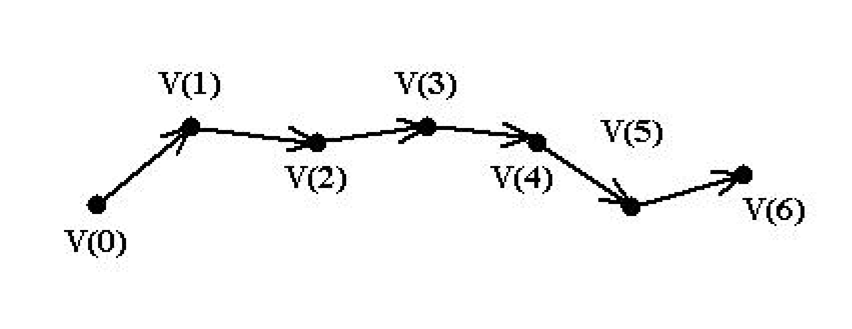
Dynamics of Hopfield Model

In contrast to feed-forward networks (wich are "static") Hopfield networks are dynamical system.

The network starts from an initial state

$$V(0) = (V_1(0), \dots ,V_n(0))^T$$

and evolves in state space following a trajectory:



Until it reaches a fixed point:

$$V(t+1) = V(t)$$

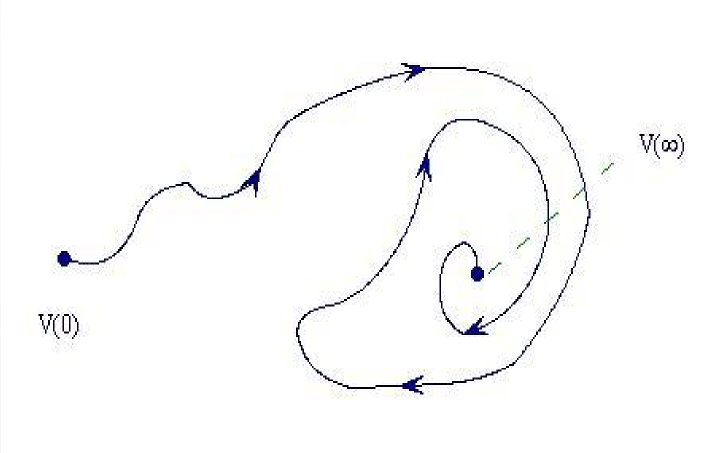
Dynamics of Hopfield Networks

What is the dynamical behavior of a Hopfield network ?

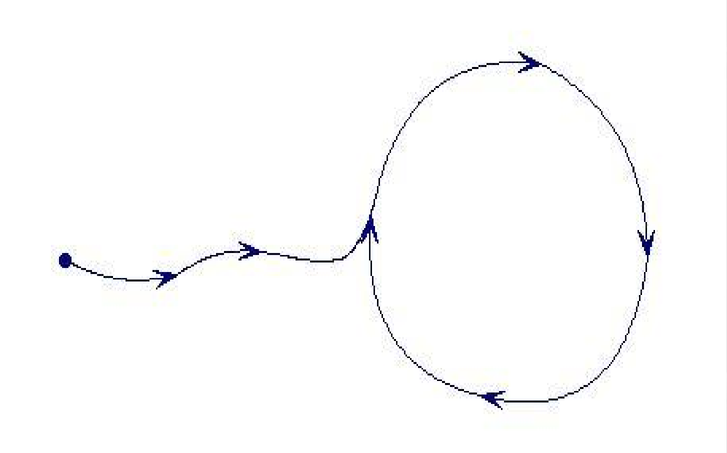
Does it coverge ?

Does it produce cycles ?

Examples



(a)



(b)

Dynamics of Hopfield Networks

To study the dynamical behavior of Hopfield networks we make the following assumption:

$$w_{ij} = w_{ji} \qquad \forall i, j = 1 \ldots n$$

In other words, if W = (w_{ij}) is the weight matrix we assume:

$$W = W^T$$

In this case the network always converges to a fixed point.
In this case the system posseses a *Liapunov* (or energy) function that is minimized as the process evolves.

The Energy Function – Discrete Case

Consider the following real function:

$$E = -\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} w_{ij}\, V_i\, V_j - \sum_{i=1}^{n} I_i\, V_i$$

and let $\Delta E = E(t+1) - E(t)$

Assuming that neuron *h* has changed its state, we have:

$$\Delta E = -\Bigg[\underbrace{\sum_{j\neq h} w_{hj}\, V_j + I_h}_{H_h}\Bigg]\Delta V_h$$

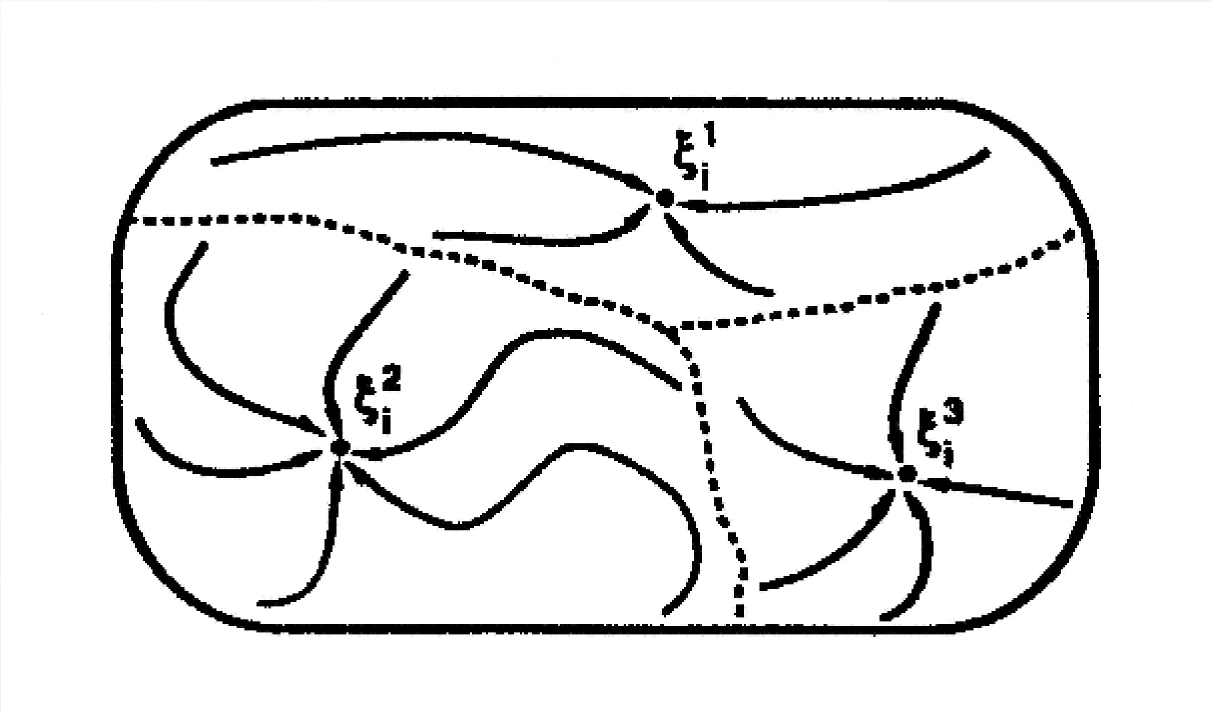
But H_h and ΔV_h have the same sign.

Hence

$$\Delta E \leq 0$$

(provided that $W = W^T$)

Schematic configuration space



model with three attractors