

On the Relations between Lumpability and Reversibility

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Abstract—In the literature devoted to the efficient solution of Continuous Time Markov Chains (CTMCs) the notions of lumpability and reversibility have a central role. In the context of lumpable Markov chains several definitions have been introduced: strong, exact and strict, just to mention a few of them. On the side of the analysis of reversible CTMCs the research community has shown great interest in the application of this notion with the aim of efficiently computing the stationary distribution of large models (e.g., obtained by composition of several processes). In this paper we show for the first time the relations between the above mentioned notions of lumpability and the concept of reversibility. The major outcome of our research is proving a strong connection between the notion of strict lumpability and that of reversibility.

I. INTRODUCTION

Performance evaluation of computer and telecommunication systems is often based on the definition and solution of a Markovian model, i.e., a model whose underlying stochastic process satisfies the Markov's memoryless property. Often the state space of the model is a denumerable set (Markov chain) and time may be continuous (CTMC) or discrete (DTMC). Among the reasons for which Markov's models became popular, an important role is played by the fact that many high-level formalisms for system modelling have been extended to include a semantics for describing the timed stochastic behaviour in a way that allows for the automatic derivation of the underlying Markov chain. Then, from the analysis of the chain one may derive the performance indices of the high-level model. Examples of formalisms with an underlying CTMC are Markovian queueing networks [1], Markovian process algebras [2], stochastic Petri nets (SPN) [3], stochastic automata networks [4], just to mention a few. The high-level specification of a Markovian model is usually compact since these formalisms allow one to use compositionality and hierarchical modelling, to a certain extent. However, the CTMC underlying a compact high level model may easily have a number of states which grows exponentially (or even faster for SPNs) with the structure of the model. This has been known since the pioneering work on queueing networks, and soon the research community investigated the possibility of exploiting some properties of the CTMC to carry out efficient exact or approximate analysis. Among the techniques that have been proposed we mention the idea of lumping a Markov chain, i.e., reducing under certain conditions its state space, that has

been introduced in [5] and that of reversibility that has been widely studied by Kelly in [6]. In particular, in Kelly's work the connections between product-forms and time-reversibility of the underlying Markov chain are investigated. Product-forms are a class of models whose stationary distribution can be expressed as the normalised product of the stationary distributions of its isolated components [7], [8].

After the original notions of lumpability (*strong* and *weak*) which have been introduced in [5] other definitions have been proposed. In particular, in [9] Takahashi uses the definition of lumpability with the aim of proposing an algorithm for the iterative aggregation and disaggregation of states in large Markov chains and in [10] the author proves that Takahashi's algorithm converges in one iteration if the CTMC is *exactly lumpable*. Based on these lumping definitions, a number of papers have been proposed with the aim of efficiently solving large Markov chains, e.g., by Sumita and Reiders [11], Buchholz [12] and Franceschinis et al. [13], just to mention a non exhaustive list of papers. In [14] the author compares different numerical approaches to the computation of the stationary distribution of CTMCs based on these research lines.

As concerns the relations between the notions of lumpability and reversibility, in [11] a lumping-based algorithm is defined whose sufficient condition for the convergence to the exact result is the time-reversibility of the Markov chains. In [15] Balsamo and Iazeolla show that the aggregation and disaggregation approach in product-form queueing networks is exact. It is worth of notice that time-reversibility is a strict condition of the CTMC, i.e., it requires that the chains $X(t)$ and $X(\tau - t)$ for $\tau \in \mathbb{R}$ are stochastically identical. Nevertheless, any ergodic Markov chain has a reversed process which is still a Markov chain and whose infinitesimal generator or probability transition matrix may be derived as studied in [16]. The notion of reversed process of non-reversible and reversible chains will play an important role in the exploration of the relations between the various definitions of lumping in the forward and in the reversed time-line.

The main outcomes of our research are the following:

- We prove that the definition of *exact* lumpability implies that of strong lumpability in the reversed process.
- We prove that a CTMC is strictly lumpable if and only if its reversed process is strictly lumpable with respect to the same partition of states.

- When a CTMC is strictly lumpable with respect to a partition, the strict lumping of its reversed process is stochastically identical to the reversed process associated with the strict lumped CTMC.
- We investigate the relations between the reversibility of a CTMC and strict lumping. In order to do so, we consider possible renamings of states (ρ -reversibility) and the reversibility of a strictly lumped process (λ -reversibility). The combination of the two definitions give what we call $\lambda\rho$ -reversibility.

Although the paper is prevalently theoretical, we strongly believe that these novel results not only represent a bridge between some areas of research (lumping, product-forms, reversible processes) but also open new perspectives for the solution of large Markov chains with iterative methods based on aggregation and disaggregation.

Structure of the paper: The paper is structured as follows. Section II introduces the notation, the definitions of lumpability, reversible and reversed CTMC. Section III studies the relations between the definitions of lumpability and those of reversed and reversible processes. In Section IV we introduce the definitions of ρ , λ and $\lambda\rho$ -reversibility and study their properties. Finally, Section V concludes the paper.

II. BACKGROUND

In this section we review some aspects of the theory of Markov processes. The arguments presented in this section apply to continuous time Markov processes with a discrete state space (CTMCs) although they can be formulated also for Discrete Time Markov Chains (DTMCs).

A. Preliminaries on Markov processes

Let $X(t)$ be a stochastic process taking values in a countable state space S for $t \in \mathbb{R}^+$. If $(X(t_1), X(t_2), \dots, X(t_n))$ has the same distribution as $(X(t_1 + \tau), X(t_2 + \tau), \dots, X(t_n + \tau))$ for all $t_1, t_2, \dots, t_n, \tau \in \mathbb{R}$ then the stochastic process $X(t)$ is said *stationary*. The stochastic process $X(t)$ is a *Markov process* if for $t_1 < t_2 < \dots < t_n < t_{n+1}$ the joint distribution of $(X(t_1), X(t_2), \dots, X(t_n), X(t_{n+1}))$ is such that

$$P(X(t_{n+1}) = i_{n+1} | X(t_1) = i_1, X(t_2) = i_2, \dots, X(t_n) = i_n)$$

is equal to

$$P(X(t_{n+1}) = i_{n+1} | X(t_n) = i_n).$$

In other words, for a Markov process its past evolution until the present state does not influence the conditional (on both past and present states) probability distribution of future behaviour. A Markov process is *time homogeneous* if the conditional probability $P(X(t+\tau) = j | X(t) = i)$ does not depend upon t , and is *irreducible* if every state in S can be reached from every other state. We assume that any CTMC with which we deal is *ergodic*. A process satisfying all these assumptions possesses an *equilibrium* (or *steady-state*) *distribution*, that is the unique collection of positive numbers π_k with $k \in S$ such that

$$\lim_{t \rightarrow \infty} P(X(t) = k | X(0) = i) = \pi_k,$$

with $\pi_k \in \mathbb{R}^+$. The transition rate between two states i and j is denoted by q_{ij} . The infinitesimal generator matrix \mathbf{Q} of a Markov process is such that the q_{ij} 's are the off-diagonal elements while the diagonal elements are formed as the negative sum of the non-diagonal elements of each row. The steady-state distribution π is the unique vector of positive numbers π_k with $k \in S$, summing to unit and satisfying the global balance equations (GBEs):

$$\pi \mathbf{Q} = \mathbf{0}.$$

Any non-trivial solution of the GBE differs by a constant but only one satisfies the normalising condition $\sum_{k \in S} \pi_k = 1$.

B. Reversibility

Given an ergodic CTMC in steady-state, $X(t)$ with $t \in \mathbb{R}^+$, we call $X(\tau - t)$ its reversed process. In the following we denote by $X^R(t)$ the reversed process of $X(t)$. It can be shown that $X^R(t)$ is also a stationary CTMC.

We say that $X(t)$ is *reversible* if it is stochastically identical to $X^R(t)$, i.e., $(X_{t_1}, \dots, X_{t_n})$ has the same distribution as $(X_{\tau-t_1}, \dots, X_{\tau-t_n})$ for all $t_1, t_2, \dots, t_n, \tau \in \mathbb{R}^+$ [6, Ch. 1].

For a stationary Markov process there exist simple necessary and sufficient conditions for reversibility expressed in terms of the equilibrium distribution π and the transition rates q_{ij} .

Proposition 1 (Transition rates of reversible processes [6]). *A stationary CTMC with state space S and infinitesimal generator \mathbf{Q} is reversible if and only if the following detailed balance equations are satisfied: for all $i, j \in S$ with $i \neq j$,*

$$\pi_i q_{ij} = \pi_j q_{ji}.$$

Clearly, a reversible CTMC $X(t)$ and its dual $X^R(t)$ have the same steady-state distribution.

An important property of reversible CTMCs is the Kolmogorov's criterion which states that the reversibility of a process can be established directly from its transition rates.

Proposition 2 (Kolmogorov's criteria [6]). *A stationary Markov process with state space S and infinitesimal generator \mathbf{Q} is reversible if and only if its transition rates satisfy the following equation: for every finite sequence of states $i_1, i_2, \dots, i_n \in S$,*

$$q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n} q_{i_n i_{n-1}} \dots q_{i_3 i_2} q_{i_2 i_1}. \quad (1)$$

C. Reversed process

The *reversed process* $X^R(t)$ of a Markov process $X(t)$ can always be defined even when $X(t)$ is not reversible. In [16] the author shows that $X^R(t)$ is a CTMC and proves that the transition rates are defined in terms of the stationary distribution of the process $X(t)$ as stated below.

Proposition 3 (Reversed process transition rates [16]). *Given the stationary Markov chain $X(t)$ with state space S and infinitesimal generator \mathbf{Q} , the transition rates of the reversed process $X^R(t)$, forming its infinitesimal generator \mathbf{Q}^R , are defined as follows:*

$$q_{ji}^R = \frac{\pi_i}{\pi_j} q_{ij}, \quad (2)$$

where q_{ji}^R denotes the transition rate from state j to i in the reversed process. The stationary distribution π is the same for both the forward and the reversed process.

In [16] the author generalises the Kolmogorov's criteria in order to encompass non-reversible CTMCs. Hereafter, for a given state i we denote by q_i (resp. q_i^R) the quantity $\sum_{j \in S, i \neq j} q_{ij}$ (resp. $\sum_{j \in S, i \neq j} q_{ij}^R$).

Proposition 4 (Kolmogorov's generalised criteria [16]). *A stationary Markov process with state space S and infinitesimal generator \mathbf{Q} has reversed process with infinitesimal generator \mathbf{Q}^R if and only if the following conditions hold:*

- 1) $q_i^R = q_i$ for every state $i \in S$;
- 2) for every finite sequence of states $i_1, i_2, \dots, i_n \in S$,

$$q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n}^R q_{i_n i_{n-1}}^R q_{i_{n-1} i_{n-2}}^R \dots q_{i_2 i_1}^R. \quad (3)$$

D. Lumpability

The concept of lumpability can be formalized in terms of equivalence relations over the state space of the Markov process. Any such equivalence induces a *partition* on the state space of the Markov chain and aggregation is achieved by aggregating equivalent states into macro-states, thus reducing the overall state space. In general, when a CTMC is aggregated the resulting stochastic process will not have the Markov property. However if the partition satisfies the so-called *strong lumpability* condition [5], [13], the property is preserved and the steady-state solution of the aggregated process may be used to derive an exact solution of the original one.

Let \sim be an equivalent relation over the state space of a CTMC. If the original state space is $\{0, 1, \dots, n\}$ then the aggregated state space is some $\{[i_0]_{\sim}, [i_1]_{\sim}, \dots, [i_N]_{\sim}\}$, where $[i]_{\sim}$ denotes the set of states that are equivalent to i and $N \leq n$, ideally $N \ll n$. In the rest of the paper, we use the following notation:

$$q_{i[k]} = \sum_{j \in [k]_{\sim}} q_{ij} \quad q_{[k]i} = \sum_{j \in [k]_{\sim}} q_{ji}.$$

By a slight abuse of notation, if no confusion arises, we simply write $[i]$ to denote the equivalence class $[i]_{\sim}$ relative to the equivalence relation \sim .

Strong lumpability has been introduced in [5] and further studied in [12], [11].

Definition 1. (Strong Lumpability) *Let $X(t)$ be a CTMC with state space $S = \{0, 1, \dots, n\}$ and \sim be an equivalence relation over S . We say that $X(t)$ is strongly lumpable with respect to \sim (resp. \sim is a strong lumpability for $X(t)$) if for any $[k] \neq [l]$ and $i, j \in [l]$,*

$$q_{i[k]} = q_{j[k]}.$$

Thus, an equivalence relation over the state space of a Markov process is a strong lumpability if it induces a partition into equivalence classes such that for any two states within an equivalence class their aggregated transition rates to any other class are the same. Notice that every Markov process is strongly lumpable with respect to the identity relation and also

the trivial relation having only one equivalence class. In [5] the authors prove that for an equivalence relation \sim over the state space of a Markov process $X(t)$, the aggregated process is a Markov process for every initial distribution if, and only if, \sim is a strong lumpability for $X(t)$. Moreover, the transition rate between two aggregated states $[i]$ and $[j]$ is equal to $q_{i[j]}$.

Let \sim be an equivalence relation over the state space of a Markov process $X(t)$. We denote by $\tilde{X}(t)$ the aggregated process with respect to \sim and by $\tilde{\mathbf{Q}}$ the corresponding infinitesimal generator.

Proposition 5. *Let $X(t)$ be a CTMC and \sim be an equivalence relation for $X(t)$. The following statements are equivalent*

- \sim is a strong lumpability for $X(t)$;
- $\tilde{X}(t)$ is a Markov process.

Moreover if \sim is a strong lumpability for $X(t)$ then for all $[i], [j] \in S/\sim$,

$$\tilde{q}_{[i][j]} = q_{i[j]}.$$

A probability distribution π is *equiprobable* with respect to a partition of the state space S of an ergodic Markov process if for all the equivalence classes $[i] \in S/\sim$ and for all $i_1, i_2 \in [i]$, $\pi_{i_1} = \pi_{i_2}$. In [10] it is introduced the notion of exact lumpability as a sufficient condition for a distribution to be equiprobable with respect to a partition.

Definition 2. (Exact Lumpability) *Let $X(t)$ be a CTMC with state space $S = \{0, 1, \dots, n\}$ and \sim be an equivalence relation over S . We say that $X(t)$ is exactly lumpable with respect to \sim (resp. \sim is an exact lumpability for $X(t)$) if for any $[k], [l] \in S/\sim$ and $i, j \in [l]$,*

$$q_{[k]i} = q_{[k]j}.$$

An equivalence relation is an exact lumpability if it induces a partition on the state space such that for any two states within an equivalence class the aggregated transition rates into such states from any other class are the same.

It is worth of notice that when we deal with the definition of exact lumpability, we assume that the CTMC has not self-loop transitions on any state. Although this seems a natural assumption when one directly works at the CTMC level, Markov chains that underlies models specified in high-level formalisms may have self-loops. Nevertheless, it is well-known that removing the self-loops from a CTMC does not change its transient or steady-state behaviour.

The following proposition states that exact lumpability induces an equiprobable distribution over its partition.

Proposition 6. *Let $X(t)$ be an ergodic CTMC with state space $S = \{0, 1, \dots, n\}$ and \sim be an equivalence relation over S . If $X(t)$ is exactly lumpable with respect to \sim (resp. \sim is an exact lumpability for $X(t)$) then for all $i_1 \sim i_2$,*

$$\pi_{i_1} = \pi_{i_2}.$$

Remark 1. *As for strong lumpability, every Markov process is exactly lumpable with respect to the identity relation. However, differently from strong lumpability, the relation*

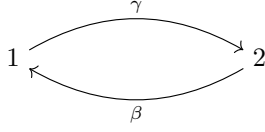


Fig. 1: A simple two state model.

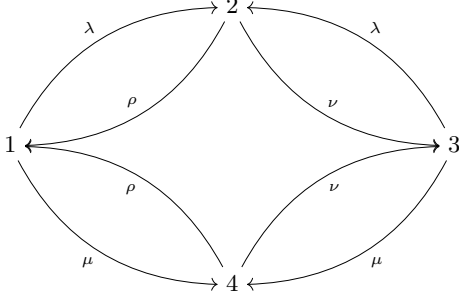


Fig. 2: A strongly, but not exactly, lumpable CTMC.

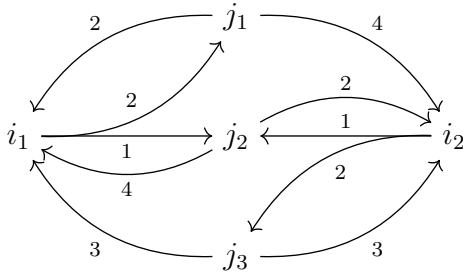


Fig. 3: A strictly lumpable CTMC.

having only one equivalence class is in general not an exact lumpability. In fact, in this case the equiprobability of its stationary distribution would not hold. This remark should clarify why we must avoid self-loops when we consider the notion of exact lumpability for CTMC. Consider the simple chain shown in Figure 1. Notice that if $\beta \neq \gamma$ then $\pi_1 \neq \pi_2$ and indeed if we consider the partition $\{1, 2\}$ then this is not exactly lumpable. However, if we consider $\beta > \gamma$ we may add a self-loop to state 2 with rate $\beta - \gamma$. Now the modified model satisfies Definition 2 but the stationary distribution would still be not equiprobable.

Finally, we introduce the notion of strict lumpability as an equivalence relation over the state space of a Markov process that is a strong lumpability with an equiprobable distribution.

Definition 3. (Strict Lumpability) Let $X(t)$ be a CTMC and \sim be an equivalence relation over its state space. We say that $X(t)$ is strictly lumpable with respect to \sim if, and only if, it is both strongly and exactly lumpable with respect to \sim (resp. \sim is a strict lumpability for $X(t)$ if, and only if, it is both a strong and an exact lumpability).

Example 1. Consider the CTMC depicted in Figure 2 with $\rho \neq \nu$. Let $S = \{1, 2, 3, 4\}$ be its state space and \sim be the equivalence relation such that $1 \sim 3$ and $2 \sim 4$, inducing the partition $S/\sim = \{\{1, 3\}, \{2, 4\}\}$. It is easy to see that \sim is a strong lumpability for $X(t)$ but it is not an exact lumpability. Indeed, for instance, $q_{\{2,4\},1} \neq q_{\{2,4\},3}$ when $\rho \neq \nu$. ■

Example 2. Consider the CTMC with state space $S = \{i_1, i_2, j_1, j_2, j_3\}$ depicted in Figure 3. Let \sim be the equivalence relation defined by: $i_1 \sim i_2$, $j_1 \sim j_2$ and $j_2 \sim j_3$. The state space S is partitioned into the following classes:

$$S/\sim = \{[i], [j]\}$$

where $[i] = \{i_1, i_2\}$ and $[j] = \{j_1, j_2, j_3\}$. Observe that

$$\begin{aligned} q_{i_1[j]} &= q_{i_2[j]} = 3 \\ q_{j_1[i]} &= q_{j_2[i]} = q_{j_3[i]} = 6 \\ q_{[i]j_1} &= q_{[i]j_2} = q_{[i]j_3} = 2 \\ q_{[i]i_1} &= q_{[i]i_2} = 0 \\ q_{[j]i_1} &= q_{[j]i_2} = 9 \\ q_{[j]j_1} &= q_{[j]j_2} = q_{[j]j_3} = 0 \end{aligned}$$

By Definitions 1 and 2, \sim is a strict lumpability for $X(t)$. ■

The next corollary follows from Propositions 5 and 6.

Corollary 1. Let $X(t)$ be an ergodic CTMC with state space S and \sim be a strict lumpability for $X(t)$. Then

- $\tilde{X}(t)$ is a Markov process;
- $\pi_{i_1} = \pi_{i_2}$, for all $i_1, i_2 \in S$ such that $i_1 \sim i_2$;
- $\tilde{q}_{[i][j]} = q_{i[j]}$, for all $[i], [j] \in S/\sim$.

We conclude this section with the following proposition that will be used in proving our results.

Proposition 7. Let $X(t)$ be a CTMC with state space S and a $\sim \subseteq S \times S$ be a strict lumpability for $X(t)$. Then, for each class $[i], [j] \in S/\sim$ it holds:

$$n_i q_{i[j]} = n_j q_{[i]j}$$

where n_h is the cardinality of the equivalence class $[h]$, with $h = i, j$.

III. LUMPABILITY AND REVERSIBILITY

In this section we prove the main result of our paper. The following theorem states that if an equivalence relation over the state space of a Markov process is an exact lumpability then it is a strong lumpability for the reversed process.

Theorem 1. Let $X(t)$ be a CTMC with state space S and $X^R(t)$ its reversed process. Let \sim be an exact lumpability for $X(t)$. Then \sim is a strong lumpability for $X^R(t)$.

In general, if \sim is a lumpability for $X(t)$ then \sim is neither a strong nor an exact lumpability for $X^R(t)$.

Example 3. Let $X(t)$ be the CTMC depicted in Figure 4 and \sim be the equivalence relation: $1 \sim 3$ and $2 \sim 4$. It is easy to prove that \sim is a strong lumpability for $X(t)$. Let us now consider the reversed process $X^R(t)$ represented in Figure 5

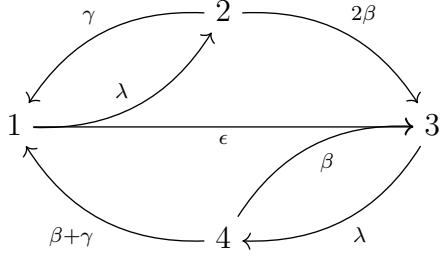


Fig. 4: A strongly but not exactly lumpable and non-reversible CTMC.

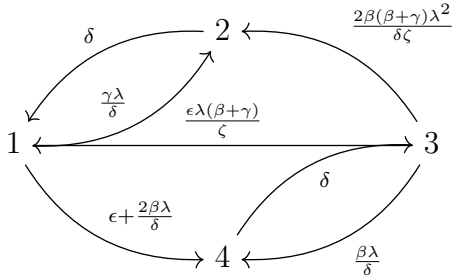


Fig. 5: Reversed process of the model in Fig. 4.

with $\delta = 2\beta + \gamma$ and $\zeta = \epsilon\gamma + 2\beta(\epsilon + \lambda)$. It holds that \sim is neither a strong nor an exact lumpability for $X^R(t)$. ■

The next theorem states that an equivalence relation is a strict lumpability for a Markov process if, and only if, it is a strict lumpability for its reversed process.

Theorem 2. Let $X(t)$ be a CTMC with state space S and $X^R(t)$ its reversed process. An equivalence relation $\sim \subseteq S \times S$ is a strict lumpability for $X(t)$ if, and only if, \sim is a strict lumpability for $X^R(t)$.

Given a stochastic process $X(t)$ with state space S , its reversed process $X^R(t)$ and an equivalence relation \sim over S , we denote by $\tilde{X}(t)$ and $\tilde{X}^R(t)$ the aggregated processes with respect to \sim corresponding to $X(t)$ and $X^R(t)$, respectively.

Corollary 2. Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be an equivalence relation. The aggregated processes $\tilde{X}(t)$ and $\tilde{X}^R(t)$ satisfy the Markov property if, and only if, \sim is a strict lumpability for $X(t)$.

If $X(t)$ is a reversible CTMC then exact lumpability is a necessary and sufficient condition for strict lumpability.

Corollary 3. Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be an equivalence relation. \sim is a strict lumpability for $X(t)$ if, and only if, \sim is an exact lumpability for $X(t)$.

We study the relationships between $\tilde{X}^R(t)$ and the reversed process of $\tilde{X}(t)$, denoted by $(\tilde{X})^R(t)$. We prove that they are stochastically identical when $X^R(t)$ has the Markov property.

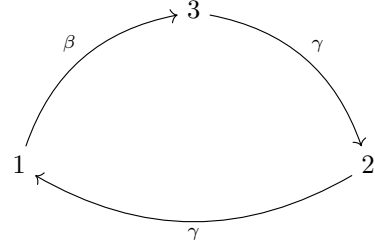


Fig. 6: A ρ -reversible CTMC.

Theorem 3. Let $X(t)$ be a CTMC with state space S and $\sim \subseteq S \times S$ be a strict lumpability for $X(t)$. Then the Markov processes $\tilde{X}^R(t)$ and $(\tilde{X})^R(t)$ are stochastically identical.

IV. LUMPABLE-BASED REVERSIBILITY

Many stochastic processes are not reversible, however the corresponding aggregated processes with respect to a lumpable relation may be reversible modulo some renaming of the state names. In this section we generalise the notion of reversibility and introduce a novel notion named $\lambda\rho$ -reversibility.

Hereafter a *renaming* ϱ over the state space of a Markov process is a bijection on S . For a Markov process $X(t)$ with state space S we denote by $\varrho(X)(t)$ the same process where state names are renamed according to ϱ . More formally, if \mathbf{Q} is the infinitesimal generator of $X(t)$ and \mathbf{Q}' is the infinitesimal generator of $\varrho(X)(t)$ it holds that: for all $i, j \in S$:

$$\begin{aligned} \pi_i &= \pi_{\varrho(i)} \\ q'_{\varrho(i)\varrho(j)} &= q_{ij}. \end{aligned}$$

We first introduce the notion of ρ -reversibility.

Definition 4 (ρ -reversibility). An ergodic CTMC with state space S is ρ -reversible if there exists a renaming ϱ on S such that $X(t)$ and $\varrho(X^R)(t)$ are stochastically identical.

Example 4. Consider the non reversible CTMC depicted in Figure 6. Observe that if we consider the renaming ϱ defined as: $\varrho(1) = 1, \varrho(2) = 3, \varrho(3) = 2$ then we can prove that $X(t)$ is ρ -reversible. ■

Another notion of reversibility named λ -reversibility and based of the concept of lumpability is introduced below.

Definition 5 (λ -reversibility). An ergodic CTMC with state space S is λ -reversible if there exists a strict lumpability \sim for $X(t)$ such that $\tilde{X}(t)$ and $\tilde{X}^R(t)$ are stochastically identical.

In other words, we say that $X(t)$ is λ -reversible with respect to an equivalence relation \sim over S if

- \sim is a strict lumpability, and
- $\tilde{X}(t)$ is reversible.

Example 5. Let $X(t)$ be the CTMC depicted in Figure 3 and \sim be the strict lumpability presented in Example 2. The state space S is partitioned into the following classes:

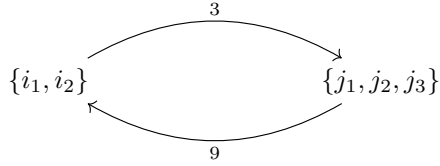


Fig. 7: The aggregated process of the CTMC in Fig. 3.

$S/ \simeq \{\{i_1, i_2\}, \{j_1, j_2, j_3\}\}$. The aggregated process $\tilde{X}(t)$, represented in Figure 7, is reversible. Hence, $X(t)$ is λ -reversible. ■

We finally introduce the notion of $\lambda\rho$ -reversibility.

Definition 6 ($\lambda\rho$ -reversibility). An ergodic CTMC with state space S is $\lambda\rho$ -reversible if there exist a strict lumpability \sim for $X(t)$ and a renaming ϱ on S/ \sim such that $\tilde{X}(t)$ and $\varrho(\tilde{X}^R)(t)$ are stochastically identical.

It is clear that a Markov process is ρ -reversible when it is $\lambda\rho$ -reversible with respect to the trivial lumpability identity. Moreover, λ -reversibility corresponds to $\lambda\rho$ -reversibility with respect to the trivial renaming identity.

By applying Proposition 1 we obtain the following necessary and sufficient condition for the $\lambda\rho$ -reversibility. We denote by $\varrho[i]$ the renaming of the class $[i]$ according to ϱ .

Proposition 8 (Transition rates of $\lambda\rho$ -reversible processes). An ergodic CTMC with state space S and infinitesimal generator \mathbf{Q} is $\lambda\rho$ -reversible if and only if there exist a strict lumpability \sim for $X(t)$ and a renaming ϱ on S/ \sim such that the following detailed balance equations are satisfied:

for all $[i], [j] \in S/ \sim$ with $[i] \neq [j]$, for all $i \in [i]$ and $j' \in \varrho[j]$:

$$\pi_{[i]} q_{i[j]} = \pi_{[j]} q_{j'[i]}$$

By applying the Kolmogorov's criterion we obtain the following characterization of lumpable reversibility.

Proposition 9. An ergodic CTMC with state space S and infinitesimal generator \mathbf{Q} is $\lambda\rho$ -reversible if and only if there exist a strict lumpability \sim for $X(t)$ and a renaming ϱ on S/ \sim such that its transition rates satisfy the following equation: for every finite sequence of states $i_1, i_2, \dots, i_n \in S$.

$$q_{i_1[i_2]} q_{i_2[i_3]} \cdots q_{i_{n-1}[i_n]} q_{i_n[i_1]} = q_{i'_1 \varrho[i_n]} q_{i'_n \varrho[i_{n-1}]} \cdots q_{i'_3 \varrho[i_2]} q_{i'_2 \varrho[i_1]}.$$

V. CONCLUSION

In this paper we have revised many notions that the literature has introduced for studying large Markov chains in an efficient way. Specifically, we have devoted our attention to the various definitions of lumpability: strong, exact and strict. Starting from the theory of reversed Markov processes developed in [6], [16], we have shown that although the exact lumping does not imply a strong lumping on CTMC $X(t)$, it implies a strong lumping in $X(\tau - t)$, i.e., its reversed process. Another important result is that a strict lumping on $X(t)$ implies a strict

lumping also in $X(\tau - t)$ (and vice versa). As a consequence, we showed that the notions of strong and strict lumpability in reversible Markov chains are equivalent. Then, we have extended the idea of reversibility by allowing a renaming of states. The intuition behind the ρ -reversibility is that $X(t)$ and $X(\tau - t)$ may be stochastically different but there exists a renaming of the states of $X(t)$ that makes it stochastically indistinguishable from $X(\tau - t)$. We have been also interested in characterising the (non-reversible) processes that have a reversible lumping (λ -reversibility). The $\lambda\rho$ -reversibility considers the processes whose lumping is reversible by means of a renaming of states. To the best of our knowledge, the paper presents new results on the theory of Markov chains. These may be exploited to define new algorithms based on the aggregation/disaggregation methods as in [9], [17] or for studying a wider class of decomposable models following the line of [18], [15].

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