Evacuating Robots from an Unknown Exit in a Disk

Evangelos Kranakis
School of Computer Science, Carleton University, Ottawa, Canada

Università Ca’Foscari Venezia, January 2015
Evacuation problem

- $k$ robots are placed inside a unit disk.
Evacuation problem

- $k$ robots are placed inside a unit disk.
- An exit is located on the perimeter.
Evacuation problem

- $k$ robots are placed inside a unit disk.
- An exit is located on the perimeter.
- Exit is placed by an adversary. Location of the exit is unknown to the robots.
Evacuation problem

- $k$ robots are placed inside a unit disk.
- An exit is located on the perimeter.
- Exit is placed by an adversary. Location of the exit is **unknown** to the robots.
- Robots can move inside the disk with unit speed.
Evacuation problem

- $k$ robots are placed inside a unit disk.
- An exit is located on the perimeter.
- Exit is placed by an adversary. Location of the exit is unknown to the robots.
- Robots can move inside the disk with unit speed.
- Evacuation time is the time when the last robot arrives at the exit.
Evacuation problem

- $k$ robots are placed inside a unit disk.
- An exit is located on the perimeter.
- Exit is placed by an adversary. Location of the exit is unknown to the robots.
- Robots can move inside the disk with unit speed.
- Evacuation time is the time when the last robot arrives at the exit.
- **Goal:** minimize the evacuation time.
Communication

- Robots collaborate in order to complete the task faster.
- Some means of communication is necessary to achieve speedup.
Robots collaborate in order to complete the task faster.
Some means of communication is necessary to achieve speedup.

Communication models

- **Wireless** – robots can exchange information at any place/time.
- **Non-wireless** (also called *face-to-face*) – robots can communicate only when located at the same point.
## Details of the model

### No visibility

- Cannot see other robots or the exit from a distance.
- In non-wireless model they become aware of other robots only when located at the same position and at the same time.
Details of the model

### No visibility
- Cannot see other robots or the exit from a distance.
- In non-wireless model they become aware of other robots only when located at the same position and at the same time.

### Memory
- Can memorize the location of the exit and go look for other robots.
**Details of the model**

<table>
<thead>
<tr>
<th>No visibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Cannot see other robots or the exit from a distance.</td>
</tr>
<tr>
<td>• In non-wireless model they become aware of other robots only when located at the same position and at the same time.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Can memorize the location of the exit and go look for other robots.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Robots share the same system of coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Robot $A$ can tell robot $B$ about the location of the exit and $B$ will be able to get there.</td>
</tr>
</tbody>
</table>
Details of the model

No visibility
- Cannot see other robots or the exit from a distance.
- In non-wireless model they become aware of other robots only when located at the same position and at the same time.

Memory
- Can memorize the location of the exit and go look for other robots.

Robots share the same system of coordinates
- Robot A can tell robot B about the location of the exit and B will be able to get there.

All robots start in the center of the disk.
- Distance to the exit is always 1.
### Details of the model

<table>
<thead>
<tr>
<th>Details</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No visibility</strong></td>
<td></td>
</tr>
</tbody>
</table>
  - Cannot see other robots or the exit from a distance.
  - In non-wireless model they become aware of other robots only when located at the same position and at the same time. |
| **Memory** |  
  - Can memorize the location of the exit and go look for other robots. |
| **Robots share the same system of coordinates** |  
  - Robot A can tell robot B about the location of the exit and B will be able to get there. |
| **All robots start in the center of the disk.** |  
  - Distance to the exit is always 1. |
| **Deterministic** |  
  - A robot that has no information about exit follows a fixed trajectory. |
Evacuation of one robot is equivalent to the cow-path problem.

**Cow-path problem**

The problem of searching for an exit point on an infinite line.
- $d$ – distance to the exit from the starting position.
- Simple doubling strategy guarantees finding the in time $9d$.
- There is no better deterministic strategy.
Variant of the cow path problem

- The doubling strategy has the problem that at the beginning it needs to make an infinite number of infinitesimal steps.
  - Not a problem if you have a “unit”.
- An interesting (and practical) modification is to introduce a cost $\delta$ for each turn.
- Then there exists strategy (very similar to simple doubling) that has only a finite number of turns and the optimal search time is $9d + 2\delta$
  
  [Demaine, Fekete, Gal 2006]

- Other variants – for example star search.
Related work – evacuation on a line

Evacuation of multiple robots on a line.

- $9d$ is optimal evacuation time of multiple robots with the same speed on a line.
- For multiple robots with different speeds $v_1 \leq v_2 \leq \ldots v_k = 1$ it is still possible to evacuate in time $9d$ as long as the minimum speed satisfies: $v_1 \geq 1/3$. [Chrobak, Gąsieniec, Gorry, SOFSEM 2015]

Exercise: Evacuation of two robots on a line.

- Two robots with same speed start at the same point on a line.
- How long does it take to evacuate in the
  - wireless model?
  - non-wireless model?
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.

$1 + 2\pi$ is also a lower bound.
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.
- On the perimeter, robot chooses a direction.
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.
- On the perimeter, robot chooses a direction.
- Robot can move inside the disk with unit speed.

$1 + 2\pi$ is also a lower bound.
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.
- On the perimeter, robot chooses a direction.
- Robot can move inside the disk with unit speed.
- Robot must traverse the whole perimeter.

$1 + 2\pi$ is also a lower bound.
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.
- On the perimeter, robot chooses a direction.
- Robot can move inside the disk with unit speed.
- Robot must traverse the whole perimeter.
- Evacuation time is $1 + 2\pi$. 

Evangelos Kranakis
Evacuating Robots from an Unknown Exit in a Disk 8/25
Single Robot: a simple example

- Robot initially placed at the centre of a unit disk.
- Robot must go to the perimeter.
- On the perimeter, robot chooses a direction.
- Robot can move inside the disk with unit speed.
- Robot must traverse the whole perimeter.
- Evacuation time is $1 + 2\pi$.
- $1 + 2\pi$ is also a lower bound.
## Results

<table>
<thead>
<tr>
<th>Communication</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-wireless</td>
<td>$k = 2$</td>
<td>$\sim 5.74$</td>
</tr>
<tr>
<td>Wireless</td>
<td>$k = 2$</td>
<td>$1 + \frac{2\pi}{3} + \sqrt{3} \sim 4.83$</td>
</tr>
<tr>
<td>Non-wireless</td>
<td>$k = 3$</td>
<td>$3 + \frac{2\pi}{3} \sim 5.09$</td>
</tr>
<tr>
<td>Wireless</td>
<td>$k = 3$</td>
<td>$\sim 4.22$</td>
</tr>
<tr>
<td>Non-wireless</td>
<td>large $k$</td>
<td>$3 + \frac{2\pi}{k}$</td>
</tr>
<tr>
<td>Wireless</td>
<td>large $k$</td>
<td>$3 + \frac{\pi}{k} + O(k^{-4/3})$</td>
</tr>
</tbody>
</table>
### $k = 2$ summary

<table>
<thead>
<tr>
<th>Communication</th>
<th>Conference</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-wireless</td>
<td>DISC 2014</td>
<td>5.744</td>
<td>5.199</td>
</tr>
<tr>
<td></td>
<td>CIAC 2015</td>
<td>5.628</td>
<td>5.255</td>
</tr>
<tr>
<td>Wireless</td>
<td>DISC 2014</td>
<td>$1 + \frac{2\pi}{3} + \sqrt{3}$</td>
<td>$1 + \frac{2\pi}{3} + \sqrt{3}$</td>
</tr>
</tbody>
</table>
Non-wireless, \( k = 2 \), algorithm

Algorithm

- Both robots go to the point \( A \).
- Robots go in the opposite directions.
- When a robot discovers the exit, it takes a chord to meet the other robot and bring it to the exit.

Evacuation time \( T = 1 + x + 2y \).

\( \alpha \) is the angle from \( B \) to \( D \), \( \alpha = 2x + y \).

\( y \) is the length of the chord on arc \( \alpha \), thus \( y = 2\sin\left(\frac{\alpha}{2}\right) \).

\( T = f(\alpha) = 1 + \frac{\alpha}{2} + 3\sin\left(\frac{\alpha}{2}\right) \).

\( \max_{\alpha} f(\alpha) \approx 5.74. \)
Non-wireless, \( k = 2 \), algorithm

**Algorithm**

- Both robots go to the point \( A \).
- Robots go in the opposite directions.
- When a robot discovers the exit, it takes a chord to meet the other robot and bring it to the exit.

Evacuation time \( T = 1 + x + 2y \).

- \( \alpha \) is the angle from \( B \) to \( D \), \( \alpha = 2x + y \).
- \( y \) is length of the cord on arc \( \alpha \), thus \( y = 2 \sin(\alpha/2) \).
- \( T = f(\alpha) = 1 + \alpha/2 + 3 \sin(\alpha/2) \).
- \( \max_{\alpha} f(\alpha) \approx 5.74 \).
Non-wireless, $k = 2$, algorithm

- Algorithm can be improved (to $\sim 5.64$) using a *strange* strategy.
Algorithm can be improved (to $\sim 5.64$) using a *strange* strategy.

Robot moves along the perimeter until point $D$ and then goes inside the disk to point $E$ hoping to meet the other robot.
Non-wireless, $k = 2$, algorithm

- Algorithm can be improved (to $\sim 5.64$) using a *strange* strategy.
- Robot moves along the perimeter until point $D$ and then goes inside the disk to point $E$ hoping to meet the other robot.
- If it does not, it returns to the perimeter and continues.
Non-wireless, $k = 2$, algorithm

- Algorithm can be improved (to $\sim 5.64$) using a *strange* strategy.
- Robot moves along the perimeter until point $D$ and then goes inside the disk to point $E$ hoping to meet the other robot.
- If it does not, it returns to the perimeter and continues.

Algorithm can be improved (to $\sim 5.628$) using a *very strange* strategy.
Theorem
It takes at least time $3 + \frac{\pi}{4} + \sqrt{2} \approx 5.199$ for two robots to evacuate in the non-wireless model.

Proof
- Within time less than $1 + \frac{\pi}{4}$ less than $\frac{\pi}{2}$ of the perimeter is explored thus there exists a square with all vertices unexplored.
  - To see this think about rotating a square inscribed inside the circle by angle $\frac{\pi}{2}$ – one of the vertices has to be explored all the time during the rotation.
- It is left to show that it takes $2 + \sqrt{2}$ to evacuate from a square with side of length $\sqrt{2}$, regardless of the positions of robots.
  - Easy case analysis.

Optimal strategy is unknown!
Wireless, $k = 2$, algorithm

**Algorithm**

- Both robots go to point $A$.
- Robots go in opposite directions along the perimeter.
- When one robot finds exit, it notifies the other using wireless model.

![Diagram](image)

Evacuation time $T(x) = 1 + x + c(x)$, where $c(x) = 2\sin(2\pi - 2x^2) = 2\sin x$.

Maximum evacuation time $\max_x T(x) = 1 + 2\pi/3 + \sqrt{3}$.

This is optimal because any algorithm after time $1 + x$ will have a chord of length $c(x)$ with both endpoints unexplored (let's see why!).
Wireless, $k = 2$, algorithm

**Algorithm**

- Both robots go to point $A$.
- Robots go in opposite directions along the perimeter.
- When one robot finds exit, it notifies the other using wireless model.

**Evacuation time**

$T(x) = 1 + x + c(x)$, where $c(x) = 2 \sin\left(\frac{2\pi - 2x}{2}\right) = 2 \sin x$.

**Maximum evacuation time**

$\max_x T(x) = 1 + \frac{2\pi}{3} + \sqrt{3}$.

This is optimal because any algorithm after time $1 + x$ will have a chord of length $c(x)$ with both endpoints unexplored (Let’s see why!).
Wireless, $k = 2$, lower bound

- It takes 1 time unit for the two robots to reach the perimeter.
- In another $2\pi / 3$ time the two robots can explore together at most $4\pi / 3$ total length of the perimeter.
- This leaves length $\geq 2\pi - 4\pi / 3 = 2\pi / 3$ of the perimeter unexplored.
- **Claim:** There is a chord of length at least $\sqrt{3}$ none of whose endpoints has been explored by a robot.

Assume no such chord exists.
It follows that for any unexplored point $A$ if we draw two chords $AA_1$ and $AA_2$ each of length $\sqrt{3}$ then each point in the arc $A_1A_2$ must be explored by a robot.
Wireless, $k = 2$, lower bound

- Same observation holds for any unexplored point in either of $\widehat{AA_1}$, $\widehat{AA_2}$.

- If we consider the leftmost and rightmost unexplored points on the circle on either side of $A$, say $B$ and $C$, then their distance must be at least $\sqrt{3}$ (if $|BC| < \sqrt{3}$ then the explored portion would exceed $4\pi/3$, a contradiction).

- Take this pair of points and consider a chord connecting them.
- Such a chord has length $\geq 2\sin(\pi/3) = \sqrt{3}$ and has both endpoints unexplored: adversary can place the exit in any of two endpoints.
- Consider the moment when some robot visits the first endpoint of the chord: the adversary places the exit in the other endpoint and such robot will have to walk at least the length of the chord.
- The total evacuation time is at least $1 + 2\pi/3 + \sqrt{3}$. 
**Large $k$ summary**

<table>
<thead>
<tr>
<th>Communication</th>
<th>Upper bound</th>
<th>Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-wireless</td>
<td>$3 + \frac{2\pi}{k}$</td>
<td>$3 + \frac{2\pi}{k} - O(k^{-2})$</td>
</tr>
<tr>
<td>Wireless</td>
<td>$3 + \frac{\pi}{k} + O(k^{-4/3})$</td>
<td>$3 + \frac{\pi}{k}$</td>
</tr>
</tbody>
</table>

Evangelos Kranakis

Evacuating Robots from an Unknown Exit in a Disk 17/25
Non-wireless, large $k$, algorithm

**Algorithm**

- Robots divide the perimeter among themselves (into arcs of size $\frac{2\pi}{k}$).
- Each robot goes to its arc, explores it, and returns to the center.
- The robot that found the exit tells the others about the location of the exit.

**Theorem**

It is possible to evacuate $k$ robots in the non-wireless model in time $3 + \frac{2\pi}{k}$.
Non-wireless, large $k$, lower bound

**Lower bound**

For any $k$ and $1 < \alpha < 2$, evacuation takes time at least 
\[
\min \left\{ 3 + \frac{\alpha \pi}{k}, 3 + 2 \sin \left( \pi - \frac{\alpha \pi}{2} \right) \right\}
\]
in the non-wireless model.
Non-wireless, large $k$, lower bound

**Lower bound**

For any $k$ and $1 < \alpha < 2$, evacuation takes time at least
\[ \min \{ 3 + \frac{\alpha\pi}{k}, 3 + 2\sin \left( \pi - \frac{\alpha\pi}{2} \right) \} \]
in the non-wireless model.

**Proof**

Take three time intervals $I_1 = [0, 1), I_2 = [1, 1 + \frac{\alpha\pi}{k}), I_3 = [1 + \frac{\alpha\pi}{k}, 3)$. Consider trajectories of "not disturbed" robots.

- **Case 1:** Trajectory of robot $r$ is in point $p$ on the perimeter during interval $I_3$. Place the exit in point $p + \pi$, antipodal to $p$. It will take time at least $3 + \frac{\alpha\pi}{k}$ for $r$ to evacuate.
Non-wireless, large $k$, lower bound

**Lower bound**

For any $k$ and $1 < \alpha < 2$, evacuation takes time at least
$$\min \left\{ 3 + \frac{\alpha\pi}{k}, 3 + 2\sin \left(\pi - \frac{\alpha\pi}{2}\right) \right\}$$
in the non-wireless model.

**Proof**

Take three time intervals $I_1 = [0, 1), I_2 = [1, 1 + \frac{\alpha\pi}{k}), I_3 = [1 + \frac{\alpha\pi}{k}, 3)$.

Consider trajectories of "not disturbed" robots.

- **Case 1**: Trajectory of robot $r$ is in point $p$ on the perimeter during interval $I_3$. Place the exit in point $p + \pi$, antipodal to $p$. It will take time at least $3 + \frac{\alpha\pi}{k}$ for $r$ to evacuate.

- **Case 2**: None of the trajectories is on the perimeter in interval $I_3$. Robots were exploring only in interval $I_2$. At time 3, $2\pi - \alpha\pi$ space on the perimeter is not visited. There exists a chord of length $2\sin(\pi - \frac{\alpha\pi}{2})$ with both endpoints not visited (Lemma about chord).
Non-wireless, large $k$, lower bound

Lemma about chord

If a total length of $x + \epsilon$ of perimeter is not explored then there exists a chord on length $2\sin(x/2)$ with both endpoints unexplored (for any $\epsilon > 0$).

Theorem

For any $k$, evacuation in the non-wireless model in the worst case takes time at least $3 + \frac{2\pi}{k} - O(k^{-2})$.

Proof.

Set $\alpha = \frac{2k}{k + 1}$ in the "Lower bound".
Wireless, large $k$, lower bound

**Theorem**

For any $k$, evacuation in the wireless model in the worst case takes time at least $3 + \frac{\pi}{k}$.

**Proof**

- Within time $1 + \frac{\pi}{k}$ less than half of the perimeter is explored.
- There exists a diameter with both endpoints unexplored.
- When a robot visits one endpoint we put the exit in the other.
- Total time of $3 + \frac{\pi}{k}$ is needed.
Algorithm

- Team $\alpha$ of size $k^{2/3}$ and $\beta$ of size $k - k^{2/3}$.
- $\alpha$ is assigned part of the perimeter of size $\pi - 2\sqrt{\pi}/k^{1/3}$.
- $\beta$ is assigned the rest.
- Teams divide their parts among themselves, explore them.
- When team $\beta$ is done, robots start moving towards the center.
- When any robot discovers the exit, it notifies others via wireless and all robots go to the exit.

Theorem

The algorithm works in time at most $3 + \frac{\pi}{k} + O(k^{-4/3})$. 
Conclusions and open problems

- There is separation between wireless and non-wireless communication:

- Difference in evacuation time becomes insignificant when $k$ gets big.

- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
  - Robots in arbitrary starting positions.
  - Arbitrary (not necessarily identical) speeds.
  - More exits.

- Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.

  - Difference in evacuation time becomes insignificant when $k$ gets big.

Some open problems:
- Tighter bounds for small $k$ ($k=2, 3, 4$).
- Robots in arbitrary starting positions.
- Arbitrary (not necessarily identical) speeds.
- More exits.

Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.
  - Difference in evacuation time becomes insignificant when $k$ gets big.

- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
  - Robots in arbitrary starting positions.
  - Arbitrary (not necessarily identical) speeds.
  - More exits.

Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.
  - Difference in evacuation time becomes insignificant when $k$ gets big.
- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
  - Robots in arbitrary starting positions.
  - Arbitrary (not necessarily identical) speeds.
  - More exits.
  - Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.
  - Difference in evacuation time becomes insignificant when $k$ gets big.

- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.
  - Difference in evacuation time becomes insignificant when $k$ gets big.
- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
  - Robots in arbitrary starting positions.
There is separation between wireless and non-wireless communication:
- Wireless helps.
- Difference in evacuation time becomes insignificant when $k$ gets big.

Some open problems:
- Tighter bounds for small $k$ ($k = 2, 3, 4$).
- Robots in arbitrary starting positions.
- Arbitrary (not necessarily identical) speeds.
Conclusions and open problems

There is separation between wireless and non-wireless communication:
- Wireless helps.
- Difference in evacuation time becomes insignificant when $k$ gets big.

Some open problems:
- Tighter bounds for small $k$ ($k = 2, 3, 4$).
- Robots in arbitrary starting positions.
- Arbitrary (not necessarily identical) speeds.
- More exits.

Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Conclusions and open problems

- There is separation between wireless and non-wireless communication:
  - Wireless helps.
  - Difference in evacuation time becomes insignificant when $k$ gets big.
- Some open problems:
  - Tighter bounds for small $k$ ($k = 2, 3, 4$).
  - Robots in arbitrary starting positions.
  - Arbitrary (not necessarily identical) speeds.
  - More exits.
  - Randomization helps significantly in the cow-path problem – it can reduce the expected search time from $9d$ to $4.59d$. Can it help in the multi-robot evacuation?
Main Published Work on Evacuation

Thank You!